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Mathematical Reviews

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HISTORY

Archibald, Raymond Clare. Outline of the History of Mathematics. The second Herbert Ellsworth Slaught Memorial Paper. Amer. Math. Monthly 56, no. 1, part II. iv+114 pp. (1949).

This is the sixth revised edition; previous editions were published in 1932, 1934, 1936, 1939, 1941.

***Lejeune, Albert. Euclide et Ptolémée. Deux Stades de l'Optique Géométrique Grecque.** Université de Louvain, Recueil de Travaux d'Histoire et de Philologie (3), no. 31, 1948. 196 pp.

The author is preparing a modern edition of Ptolemy's "Optics," which is only preserved in a Latin translation from the Arabic, published by Govi in 1885. The present work studies the development of Greek optics from the early phase of geometrical optics to the investigation of binocular vision. While Euclid's optics is hardly more than a theory of the fundamental laws of perspective we find in Ptolemy experiments which have become classical for physiological optics. The discussion of the theory of reflection and refraction will be published in a subsequent paper.

O. Neugebauer (Providence, R. I.).

***Cohen, Morris R., and Drabkin, I. E. A Source Book in Greek Science.** McGraw-Hill Book Company, Inc., New York, N. Y., 1948. xxii+579 pp.

About one half of this work is devoted to mathematics, mechanics, and its applications whereas the remaining parts deal with biology and the medical sciences. Following the general plan of this series, selections of important original texts are given in translation and commentaries are added to facilitate the reading. The choice of material, made by the editors, seems to the reviewer very fortunate. Instead of repeating too many easily available passages from Euclid, Archimedes, and Apollonius, they have presented many sections from works on astronomy, geography, mechanics, and optics, which are very little known but nevertheless very important for the understanding of Greek science and thus for its development during the Middle Ages.

O. Neugebauer (Providence, R. I.).

***Enriques, Federigo, e Mazziotti, Manlio. Le Dottrine di Democrito d'Abdera. Testi e Commenti.** Nicola Zanichelli, Bologna, 1948. xxiii+339 pp.

Arranged according to subjects we find here a translation of the fragments of Democritus's writings as preserved through later references and collected by Diels. Each section is preceded by an introduction of the editors with references to the modern literature. *O. Neugebauer.*

Bruins, E. M. Some remarks on ancient calculation. Nederl. Akad. Wetensch., Proc. 52, 161–163 = Indagationes Math. 11, 50–52 (1949).

Supplementary remarks to a previous article [same Proc. 51, 332–341 = Indagationes Math. 10, 121–130 (1948); these Rev. 9, 483]. *O. Neugebauer* (Providence, R. I.).

***Thorndike, Lynn. The Sphere of Sacrobosco and Its Commentators.** The University of Chicago Press, Chicago, Ill., 1949. x+496 pp. $10.00.

"In this volume is presented a text based primarily upon manuscripts, although with constant reference to the early printed editions and to quotations by early commentators, of the Tractatus de sphera of Iohannes de Sacrobosco, which was the clearest, most elementary, and most used textbook in astronomy and cosmography from the thirteenth to the seventeenth century. There is also offered an English translation of the Sphere from the foregoing Latin text. Next is printed for the first time, in a Latin text based upon six manuscripts and in an English translation thereof, the commentary on the Sphere composed by Robertus Anglicus in 1271. Then are reproduced in Latin the commentary attributed to Michael Scot and that by Cecco d'Ascoli of the early fourteenth century. . . . Another text, published here for the first time, consists of anonymous marginal glosses and comments—apparently drawn in some cases from Michael Scot and Robertus Anglicus—which accompany the Sphere in two of our earlier manuscripts. . . . Briefer extracts from other as yet unpublished thirteenth-century treatises on the sphere of the universe and commentaries on the Sphere of Sacrobosco are added in appendixes. One is perhaps by Andalò di Negro and of the early fourteenth century." The introduction [75 pp.] discusses the origin and later history of Sacrobosco's work and contains a great amount of information about the relations between Arabic and Western astronomy in the Middle Ages.

O. Neugebauer (Providence, R. I.).

Neugebauer, O. Mathematical methods in ancient astronomy. Bull. Amer. Math. Soc. 54, 1013–1041 (1948).

An invited address to the American Mathematical Society.

Zinner, E. Über die früheste Form des Astrolabs. Naturf. Ges. Bamberg. Ber. 30, 9–21 (1947).

Fujiwara, M. Miscellaneous notes on the history of Wazan. VII. (The works of Takakazu Seki). Tôhoku Math. J. 48, 201–214 (1941). (Japanese)

For note VI cf. same J. 47, 322–338 (1940); these Rev. 2, 306.

Natucci, A. Area del segmento circolare e volume del tetraedro in Cina. Period. Mat. (4) 26, 153–156 (1948).

de Vaux, Carra. Une solution arabe du problème des carrés magiques. Rev. Hist. Sci. Appl. 1, 206–212 (1948).

Coolidge, J. L. The story of the binomial theorem. Amer. Math. Monthly 56, 147–157 (1949).

Murnaghan, F. D. The evolution of the concept of number. Scientific Monthly 68, 262–269 (1949).

de Vries, Hk. Historical studies. XXV. On the infinite and the imaginary, or "surrealism" in mathematics. Nieuw Tijdschr. Wiskunde **36**, 82–96, 115–121 (1949). (Dutch)

Boyer, C. B. Clairaut and the origin of the distance formula. Amer. Math. Monthly **55**, 556–557 (1948).

Higgins, Thomas James. History of the operational calculus as used in electric circuit analysis. Elec. Engrg. **68**, 42–45 (1949).

Collet, Claude-Georges, et Itard, Jean. Un mathématicien humaniste, Claude-Gaspar Bachet de Méziriac (1581–1638). Rev. Hist. Sci. Appl. **1**, 26–50 (1947).

***Candido, Giacomo. Scritti Matematici.** Edited by Enea Bortolotti and Enrico Nannei. Casa Editrice Marzocco, Firenze, 1948. xv+802 pp. (1 plate).

Obituary: N. G. Čebotarëv. Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 68–71 (1947). (Russian)

Amato, V. Obituary: Michele Cipolla. Matematiche, Catania **3**, iii–xvi (1 plate) (1948).

Mignosi, Gaspare. Obituary: Michele Cipolla. Ann. Mat. Pura Appl. (4) **26**, 217–220 (1947).

Wiener, Norbert. Obituary: Godfrey Harold Hardy (1877–1947). Bull. Amer. Math. Soc. **55**, 72–77 (1949).

Obituary: Aleksandr Mihaĭlovič Lyapunov (1857–1918). Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 467–468 (1948). (Russian)

Smirnov, V. I. Outline of the life of A. M. Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 469–478 (1948). (Russian)

Smirnov, V. I. Survey of the scientific work of A. M. Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 479–552 (1948). (Russian)

List of the publications of A. M. Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 553–560 (1948). (Russian)

Stepanov, V. V., and Kalinin, S. V. Aleksandr Mihaĭlovič Lyapunov: A brief survey of his life and scientific work. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 161–167 (1949). (Russian)

Schlapp, Robert. Colin Maclaurin: A biographical note. Edinburgh Math. Notes **37**, 1–6 (1949).

Sergescu, P. Le centenaire du Père M. Mersenne. Rev. Gén. Sci. Pures Appl. N.S. **55**, 193–195 (1948).

Sergescu, P. Mersenne l'animateur (8 septembre 1588–1er septembre 1648). Rev. Hist. Sci. Appl. **2**, 5–12 (1948).

de Waard, C. A la recherche de la correspondance de Mersenne. Rev. Hist. Sci. Appl. **2**, 13–28 (1948).

Littlewood, J. E. Newton and the attraction of a sphere. Math. Gaz. **32**, 179–181 (1948).

Boyer, C. B. Newton as an originator of polar coördinates. Amer. Math. Monthly **56**, 73–78 (1949).

Conte, Luigi. Sul modo di mettere in equazione le questioni geometriche. (Dall' "Arithmetica Universalis" di I. Newton.) III. Period. Mat. (4) **26**, 133–152 (1948).

For parts I and II cf. Period. Mat. (4) **25**, 1–15, 165–180 (1947); these Rev. **9**, 169, 486.

Archibald, R. C. Bartholomäus Pitiscus (1561–1613). Math. Tables and Other Aids to Computation **3**, 390–397 (1949).

Milne, E. A., and White, F. Puryer. Obituary: Herbert William Richmond. J. London Math. Soc. **24**, 68–80 (1949)—Obit. Notices Roy. Soc. London **6**, no. 17, 219–230 (1948).

Bernšteĭn, S. N., and Giršval'd, L. Ya. Obituary: D. M. Sincov. Uspehi Matem. Nauk (N.S.) **2**, no. 4(20), 191–206 (1947). (Russian)

Fleckenstein, Joachim Otto. Pierre Varignon und die mathematischen Wissenschaften im Zeitalter des Cartesianismus. Arch. Internat. Hist. Sci. **28**, 76–138 (1948).

Kuz'min, R. O. The life and scientific activity of Egor Ivanovič Zolotarëv. Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 21–51 (1947). (Russian)

FOUNDATIONS

***Boole, George. The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning.** Philosophical Library, New York, N. Y., 1948. iv+82 pp.

Photographic reprint of the original edition [Cambridge, 1847].

Rosser, J. B., and Turquette, A. R. Axiom schemes for *M*-valued functional calculi of first order. I. Definition of axiom schemes and proof of plausibility. J. Symbolic Logic **13**, 177–192 (1948).

Die Methode der Verf. m-vertige Aussagenkalküle zu axiomatisieren [J. Symbolic Logic **10**, 61–82 (1945); diese Rev. **7**, 185] wird auf Funktionenkalküle ausgedehnt. Unter den m Wahrheitswerten $1, \cdots, m$ seien $1, \cdots, s$ "ausgezeichnet." Jede Wahrheitsfunction $F(P_1, \cdots, P_r)$ kann beschrieben werden mithilfe von "und," "oder" und den "Grundfunktionen" $T_k(P)$, die dadurch definiert sind, dass $T_k(P)$ genau dann "ausgezeichnet" ist, wenn P den Wert K hat. Mit Negation und Alloperator lassen sich auch quantifizierte Wahrheitsfunktionen beschreiben. Zur vollständigen Axiomatisierung werden solche Axiome gewählt, dass für jede Wahrheitsfunktion F und jeden Wert K die "Äquivalenz" mit einer "Normalform" $N_k(f)$ (die aus der Beschreibung entsteht) ableitbar ist; und dass alle (ausgezeichneten) "Normalformen" ableitbar sind. Zum Schluss wird bewiesen dass alle ableitbaren Formeln ausgezeichnet sind.

P. Lorenzen (Bonn).

Markov, A. A. On the dependence of axiom B6 on the other axioms of the Bernays-Gödel system. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 569–570 (1948). (Russian)

Simplification of the system of axioms for set-theory given by Bernays [J. Symbolic Logic **2**, 65–77 (1937)] and modified by Gödel [The Consistency of the Continuum Hypothesis, Princeton University Press, Princeton, N. J., 1940; these Rev. **2**, 66]. *A. Heyting* (Amsterdam).

Church, Alonzo. Conditioned disjunction as a primitive connective for the propositional calculus. Portugaliae Math. **7**, 87–90 (1948).

Es wird ein vollständiges, unabhängendes und selbstduales System von Wahrheitsfunktionen angegeben: t='das Wahre,' f='das Falsche,' $[p, q, r]=(p \vee \bar{q}) \wedge (q \vee r)$='bedingte Disjunktion.' Die üblichen Systeme: 'Konjunktion, Disjunktion, Negation' bzw. 'Konjunktion, Implikation, das Falsche' sind abhängig bzw. nicht selbst-dual. Gegenüber dem System 'Implikation ($\bar{p} \vee q$), duale Implikation ($p \wedge \bar{q}$)' besteht der Vorteil einer selbst-dualen Definition von 'Position' und 'Negation': $p=[t, p, f]$, $\bar{p}=[f, p, t]$. *P. Lorenzen* (Bonn).

Suszko, Roman. Concerning logic without axioms. Kwartalnik Filozoficzny **17**, 199–205, 319–320 (1948). (Polish. English summary)

In a formal system axioms are usually obtained by substitution in some theorems of propositional calculus. These axioms do not play an essential part in the system but are necessary as means of inference. The problem of the paper is their elimination by accepting instead certain rules of inference. The author is searching only for rules in which no premiss nor the conclusion is tautologically true. The theorem he proves states: it is sufficient to accept the following rules (1) p, $Cpq \rightarrow q$; (2) $q \rightarrow Cpq$; (3) $CCpqr \rightarrow Cqr$; (4) $CpCqr \rightarrow CCpqCpr$; (5) $Cpq \rightarrow CCrpCCqsCCrs$; (6) $CCpqr \rightarrow CNpr$; (7) Cpq, $CNpq \rightarrow q$. *H. Hiż.*

Krokiewicz, Adam. Sur la logique stoïcienne. Kwartalnik Filozoficzny **17**, 173–197, 319 (1948). (Polish. French summary)

The logic of the Stoics was a logic of propositions, in opposition to Aristotelian class-logic. The Stoics had a clear idea of this difference and of the logical priority of propositional logic over class-logic. Their main argument was that the reduction to the first figure of the syllogisms such as Baroco requires laws of propositional logic. This paper follows the suggestions of Łukasiewicz [Erkenntnis **5**, 111–131 (1935)] and describes how the Stoics built their logic. They accepted five fundamental rules of inference: (1) Cpq, $p \rightarrow q$; (2) Cpq, $Nq \rightarrow Np$; (3) $NKpq$, $p \rightarrow Nq$; (4) Dpq, $p \rightarrow Nq$; (5) Dpq, $Nq \rightarrow p$. The functor 'D' here represents disjunction which should rather be called 'nonequivalence.' Many other rules were known and reduced to these five which were axiomatically accepted. They understood that a rule of inference is valid or invalid, whereas a logical statement is true or false. They also knew how to transform a valid rule into a true statement. In addition they also accepted so-called themes. It is not clear what they meant by them. Only two (out of four) of axiomatically accepted themes are known to us and even these only on second-hand information. The author suggests the hypothesis that they were logical statements, i.e., statements of the object language. In the second part of the paper the author analyses the laws of propositional logic which were known or used before being systematized by the Stoics. *H. Hiż* (Warsaw).

Zawirski, Zygmunt. Origin and development of intuitionistic logic. Kwartalnik Filozoficzny **16**, 165–222 (1946). (Polish)

Mostowski, Andrzej. Proofs of non-deducibility in intuitionistic functional calculus. J. Symbolic Logic **13**, 204–207 (1948).

The relations between the intuitionistic calculus of propositions and Brouwerian algebra, established by McKinsey and Tarski [same vol., 1–15 (1948); these Rev. **9**, 486] are extended to the intuitionistic functional calculus F. Let L be a complete Brouwerian lattice [G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940; these Rev. **1**, 325], and I any nonvoid set. Individual variables of F, $\vee$, $\wedge$, $\rightarrow$ and $\neg$ being interpreted as by McKinsey and Tarski, a functional variable in F with k arguments is interpreted as a variable running through the functions with k arguments in I whose values are in L. The operations $(\exists x)$ and (x) in F are interpreted as the lattice-operations S_x (upper bound of a subset of L) and its dual P_x. Now if A is deducible in F, the corresponding (I, L)-functional vanishes identically for every I and L. As an application, the nondeducibility in F is proved of $(x)[a \vee b(x)] \rightarrow [a \vee (x)b(x)]$. *A. Heyting.*

Brouwer, L. E. J. Essentially negative properties. Nederl. Akad. Wetensch., Proc. **51**, 963–964 = Indagationes Math. **10**, 322–323 (1948). (Dutch)

Let A be an assertion such that neither the absurdity nor the absurdity of the absurdity of A has been proved. Then an infinitely continuing sequence of rational numbers $a_1, a_2, \cdots$ can be created according to the following directions. As long as during the choice of the a_n the creating subject has experienced neither the truth nor the absurdity of A, each a_n is chosen equal to 0. But as soon as between the choice of a_{r-1} and that of a_r the creating subject has experienced the truth of A, then a_m is chosen equal to a 2^{-r} for every $m \geqq r$; as soon as between the choice of a_{s-1} and that of a_s the creating subject has experienced the absurdity of A, then a_p is chosen equal to -2^{-s} for every $p \geqq s$. This sequence converges positively to a real number p, for which $p=0$ is contradictory, whereas neither $p>0$ nor $p<0$ is known. *A. Heyting* (Amsterdam).

Brouwer, L. E. J. Remarks on the principle of the excluded third and on negative assertion. Nederl. Akad. Wetensch., Proc. **51**, 1239–1243 = Indagationes Math. **10**, 383–387 (1948). (Dutch)

The following assertions are compared as to their fields of validity. (I) (simple principle of excluded third): Every assignment τ of a property to a mathematical entity can be judged, i.e., either proved or reduced to absurdity. (II) (complete principle of the excluded third): If a, b and c are species of mathematical entities, if further both a and b form part of c, and if b consists of those elements of c which cannot belong to a, then c is identical with the union of a and b. (III) If for an assignment τ of a property to a mathematical entity the noncontradictoriness has been established, the truth of τ can be demonstrated likewise. (IV) (principle of reciprocity of complementarity): If a, b and c are species of mathematical entities, if further a and b

form part of c, and if b consists of the elements of c which cannot belong to a, then a consists of the elements of c which cannot belong to b. (V) (simple principle of testability): Every assignment τ of a property to a mathematical entity can be tested, i.e., proved to be either noncontradictory or absurd. (VI) (complete principle of testability): If a, b, d and c are species of mathematical entities, if each of the species a, b and d forms part of c, if b consists of the elements of c which cannot belong to a, and d of the elements of c which cannot belong to b, then c is identical with the union of b and d.

For a single assertion τ, (I) is noncontradictory, but the simultaneous enunciation of the principle for all elements of a species of assertions does in some cases lead to a contradiction; thus (II) is contradictory. Assertions (III) and (V) are consequences of (I); (IV) and (VI) are consequences of (II). For negative assertions (III) is valid and (V) is equivalent to (I). The field of validity of (I) is identical with the intersection of the fields of validity of (V) and (III). Examples are given of assertions for which (V), but not (III); or (III), but not (V), is valid. Absurdity, just as truth, is a universally additive property of mathematical assertions, i.e., if it holds for each element α of a species of assertions, it also holds for the assertion which is the union of the assertions α. Noncontradictoriness is not universally additive, but it possesses the property of finite additivity. [A translation of this paper, with some additions, is contained in pp. 1244–1249 of the paper reviewed below.]

A. Heyting (Amsterdam).

✶Brouwer, L. E. J. Consciousness, philosophy, and mathematics. Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 1235–1249 (1949).

By means of an introspective psychological method, the author sketches "the phases consciousness has to pass through in its transition from its deepest home to the exterior world in which we cooperate and seek mutual understanding." Among observations of a more general nature, the author indicates the position of intuitionistic mathematics. The latter is regarded as a pre-linguistic activity based on the distinction of past and present in consciousness. Logic is characterized as a system of rules of language which have been recognized in the past usually to lead from expressions representing truths to other expressions of the same status. The establishment of truth, it is argued, depends upon a type of experience more basic than logical derivation. As an example of the unreliability of logic, the author cites in particular the principle of excluded middle. This receives a simple formulation in terms of a property of a single mathematical entity and a complete formulation in terms of species. Cases are discussed in which the principle and corollaries are applicable. Statements to which various of these classical principles do not apply are constructed by means of convergent sequences of real numbers using an unspecified statement which is, at the time of the construction, not known to be either absurd or not absurd. [Cf. the preceding review.] *D. Nelson.*

✶de Iongh, J. J. Restricted forms of intuitionistic mathematics. Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 744–748 (1949).

After a brief review of the problems of a constructive interpretation of intuitionistic logic, the author suggests that the intuitionistic dichotomy between pre-linguistic mathematical constructions and the linguistic expression of these constructions is not justified; there exists, rather, an interdependence, and we may expect formal systems to admit degrees of constructive interpretation depending upon the extent to which the linguistic element enters. The significist position, which the author advocates, finds the principal value of intuitionistic logic in the distinction which it allows among concepts of various constructive levels. He cites as example nine intuitionistically distinct prefixes made up of quantifiers and negation symbols corresponding to the one classical prefix $(x)(Ey)(z)$ used in defining convergence of a sequence of real numbers. *D. Nelson.*

de Bengy-Puyvallée, Renaud. Sur la relation de composabilité dans les logiques de complémentarité. C. R. Acad. Sci. Paris **228**, 624–626 (1949).

Griss [same C. R. **227**, 946–948 (1948); these Rev. **10**, 277] proposed that the intersection of two sets can be constructively defined only in case it is not null. The author notes this as an analogy to the situation in quantum mechanics in which the compatibility of two observations p and q obtains only in case, in a phase space, the intersection of linear subspaces corresponding to p and q has nonzero dimension. To describe these two situations, the author suggests analogous interpretations of the relation of composability [same C. R. **220**, 589–591 (1945); **226**, 454–456 (1948); these Rev. **7**, 186; **9**, 322]. The development of these interpretations will, of course, depend upon a complete formulation of the logic of composability; see the review by Bernays [J. Symbolic Logic **12**, 133–134 (1947)] of a paper by Destouches-Février. There would appear to be some difficulty in the author's identification of the composability of two observations p and q with the product of constructions in the sense of Kolmogoroff [Math. Z. **35**, 58–65 (1932)], since indicated constructions corresponding to both p and q may be effected when they are not composable. [Erratum: p. 625, insert "C" before "[b" in the last displayed formula.] *D. Nelson* (Washington, D. C.).

✶Popper, K. R. The trivialization of mathematical logic. Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 722–727; errata, p. 1259 (1949).

This is a summary, without proofs or even the details of certain definitions, of the author's program for deriving the whole of mathematical logic from definitions of the "formatic signs." Parts of this program have appeared previously [Mind **56**, 193–235 (1947); **57**, 69–70 (1948); Proc. Aristotelian Soc. N.S. **47**, 251–292 (1947); Nederl. Akad. Wetensch., Proc. **50**, 1214–1224 (1947); **51**, 173–183, 322–331 (1948) = Indagationes Math. **9**, 561–571 (1947); **10**, 44–54, 111–120 (1948); these Rev. **9**, 130, 486, 321, 486, 487]; this paper appears to go beyond the previous papers only in certain minor points regarding identity and Aristotelian logic. The whole program is very obscure, and has not been without serious error [see these Rev. **9**, 321]; likewise it was anticipated, in many respects, by the work of Gentzen and others [see the cited reviews].

H. B. Curry (State College, Pa.).

✶Fiala, F. Sur le caractère dialectique de la notion de différentielle. Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 705–707 (1949).

***Apéry, R. Axiomes et postulats.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 708–710 (1949).

***Alexits, G., et Fenyö, E. Les fondements des mathématiques et la philosophie du matérialisme dialectique.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 711–712 (1949).

***Riabouchinsky, D. Réhabilitation du recours à l'intuition sensible en analyse mathématique.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 713–714 (1949).

***Van Hagens, B. L'instrument mathématique au service de l'homme dans la connaissance de la nature.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 715–716 (1949).

***Lorenzen, P. The overcoming of logical relativism.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, p. 717 (1949).

***Virieux-Reymond, A. La logique stoïcienne.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 718–719 (1949).

***Dürr, K. Les diagrammes logiques de Leonhard Euler et de John Venn.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 720–721 (1949).

***Feys, Robert. L'abstraction en logique formalisée.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 731–734 (1949).

***Dopp, J. La notion d'existence dans la logique moderne.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 735–739 (1949).

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***Fraenkel, A. A. The relation of equality in deductive systems.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 752–755 (1949).

***Kalmár, L. On unsolvable mathematical problems.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 756–758 (1949).

***Mostowski, A. Sur l'interpretation géométrique et topologique des notions logiques.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 767–769 (1949).

***Curry, H. B. Languages and formal systems.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 770–772 (1949).

***Kokoszyńska, Maria. On a certain condition of a semantical theory of science.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 773–775 (1949).

***Grzegorczyk, A. Un essai d'établir la sémantique du langage descriptif.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 776–778 (1949).

***Britton, K. What is a rule of language?** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 779–781 (1949).

***Rynin, D. Meaning and formation rules.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 782–784 (1949).

***Saarnio, U. Der Begriff der Hierarchie und die logischen Paradoxien.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 785–790 (1949).

***von Wright, Georg Henrik. On confirmation.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 794–796 (1949).

***Servien, P. Probabilités, erreurs, quanta.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 797–799 (1949).

***Hadamard, Jacques. An Essay on the Psychology of Invention in the Mathematical Field.** Princeton University Press, Princeton, N. J., 1949. xiii+145 pp. $2.50.

This is a revised edition; the original edition appeared in 1945; cf. these Rev. 6, 198.

Bouligand, G. Sur une doctrine de la connaissance mathématique et ses incidences historiques. Arch. Internat. Hist. Sci. 2, 291–302 (1949).

Bouligand, Georges. Libres vues sur la connaissance mathématique. Rev. Questions Sci. (5) 10, 5–11 (1949).

ALGEBRA

de Bruijn, N. G., and Erdös, P. On a combinatorial problem. Nederl. Akad. Wetensch., Proc. **51**, 1277–1279 = Indagationes Math. **10**, 421–423 (1948).

Suppose $A_1, \cdots, A_m$ are subsets of the elements $a_1, \cdots, a_n$, and that no A contains all the a's. Suppose also that each pair (a_i, a_j) occurs in one and only one A. It is proved here that $m \geqq n$, and that equality may occur only if (1) the system is of the type $A_1 = (a_1, \cdots, a_{n-1})$, $A_2 = (a_1, a_n)$, $\cdots$, $A_n = (a_{n-1}, a_n)$ or if (2) n is of the form $n = k(k-1)+1$ and all the A's have k elements, and each a occurs in exactly k of the A's. The authors remark that a finite projective plane with k points on a line satisfies the second possibility, and in fact their proof shows this to be the only possibility. Nevertheless they assert that F. W. Levi [Finite Geometrical Systems, University of Calcutta, 1942; these Rev. **4**, 50] has "constructed a nonprojective example with $k=9$." They also give Gallai's proof of the theorem that, given n points in the real projective plane not all on a line, there exists a line which goes through exactly two of them. *M. Hall, Jr.*

✱Dörrie, Heinrich. Kubische und biquadratische Gleichungen. Leibniz Verlag, München, 1948. 260 pp. 24 DM.

There is no doubt that the author has achieved his aim to show that the subjects of cubic and biquadratic equations are neither too uninteresting nor too unimportant to be included in courses in higher schools. The treatment is elementary, not based on abstract ideas and only a few connections with the theory of general algebraic equations are pointed out. This book contains, however, a valuable collection of material, including a chapter on Diophantine equations. Applications of the theory to two and three dimensional analytical geometry of conics and quadrics and of plane cubic curves, and to problems in mechanics, optics and in circuit theory, are discussed in detail.
O. Todd-Taussky (London).

Selmer, Ernst S. A remark on cyclic determinants (circulants). Norsk Mat. Tidsskr. **30**, 97–107 (1949). (Norwegian)

A discussion of various properties of cyclic determinants, largely expository. The author develops a method for the determination of the inverse matrix in a form suitable for numerical computations. *O. Ore* (New Haven, Conn.).

Parker, W. V. On the characteristic equations of certain matrices. Bull. Amer. Math. Soc. **55**, 115–116 (1949).

If $ACA=0$, then AB and $A(B+C)$ have the same characteristic equation, where A is an m by n matrix and B and C are n by m. The condition $ACA=0$ is equivalent to $C=C_1+C_2$ where $AC_2=C_1A=0$. *W. Givens.*

Campedelli, Luigi. Una dimostrazione geometrica delle proprietà dell'equazione secolare generalizzata. Period. Mat. (4) **26**, 74–80 (1948).

Negri, Domenico. Risoluzione di sistemi di 3 equazioni di 2° grado in 3 incognite. Boll. Un. Mat. Ital. (3) **3**, 161–167 (1948).

The solution of a system of 3 quadratics in 3 unknowns as described in Serret [Cours d'Algèbre Supérieure, Paris, 1877] is discussed, the object being to simplify the treatment and complete the proofs. *O. Todd-Taussky.*

Fiore, Maria. A proposito di alcune disequazioni lineari. Rend. Sem. Mat. Univ. Padova **17**, 1–8 (1948).

Let a_{ij} be real and b_i nonnegative for $i, j=1, 2, \cdots, n$. Let the quadratic form with coefficients $(a_{ij}+a_{ji})/2$ be nonnegative for all real values of the indeterminates. The paper proves the existence of bounds K_i such that $K_i \leqq x_i$ for every solution of $x_i \leqq 0$, $0 \leqq a_{i1}x_1 + \cdots + a_{in}x_n + b_i$ $(i=1, 2, \cdots, n)$.
J. M. Thomas (Durham, N. C.).

Hua, Loo-Keng. Geometry of symmetric matrices over any field with characteristic other than two. Ann. of Math. (2) **50**, 8–31 (1949).

The paper, while relatively self-contained, is concerned with a sharpening of results obtained earlier. [See Trans. Amer. Math. Soc. **57**, 441–481, 482–490 (1945); **61**, 193–228, 229–255 (1947); these Rev. **7**, 58; **9**, 171. Compare also C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 95–97, 195–196 (1946); these Rev. **8**, 328.] It is now proved that, if Φ is any field not of characteristic two and Φ_2 is the multiplicative group of all square elements of Φ, then every transformation carrying the projective space of symmetric matrices of order $n>1$ onto itself and keeping arithmetic distance invariant is of the form $(X_1, Y_1) = Q(X^\sigma, Y^\sigma)\mathfrak{T}$, where Q is a nonsingular n by n matrix, σ is an automorphism of Φ, $\mathfrak{T}$ is a $2n$ by $2n$ matrix satisfying

$$\mathfrak{T}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\mathfrak{T}' = a\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and a runs over a complete residue system of Φ/Φ_2. Previous conditions on "continuity," "sense" and "harmonic separation" are eliminated and the restriction to the complex field is dropped. The method involves a detailed study of the case $n=2$ with a classification of involutions in the geometry and a construction analogous to that of the complete quadrilateral in ordinary projective geometry.
W. Givens (Knoxville, Tenn.).

Todd, J. A. Ternary quadratic types. Philos. Trans. Roy. Soc. London. Ser. A. **241**, 399–456 (1948).

Complete systems of concomitants for 2, 3, 4 ternary quadratic forms were given by Gordan [Math. Ann. **20**, 487–515 (1882)], Ciamberlini [Giorn. Mat. Battaglini (1) **24**, 141–157 (1886)] and Turnbull [Proc. London Math. Soc. (2) **9**, 81–121 (1910)]. However, by Peano's theorem the concomitants of a system of any number of ternary quadratics can be expressed in terms of the concomitants of 5 or fewer, together with the alternating linear invariant of 6 quadratics. In this paper the author obtains a complete system for 5 quadratics, which thus leads to a complete knowledge of the concomitant types of any number of quadratics. The bulk of the paper consists of an analysis by classical methods, utilizing Turnbull's contracted symbolic notation, of all the concomitant forms which arise from 5 ternary quadratics, and a complete system of 176 forms is obtained. Though proving that the system is complete, this method gives no guarantee that all the forms are irreducible. However, the last section of the paper approaches the problem in a different way, using S-functional analysis. This gives a system which is certainly irreducible, but gives no information as to its completeness. But this method also gives a system of 176 forms. The two methods taken together therefore demonstrate that the system is both complete and irreducible. *D. E. Littlewood.*

Papy, Georges, et Tournay, Francis. Classification des formes cubiques alternées à six indéterminées par rapport au groupe canonique de degré trois. Bull. Soc. Roy. Sci. Liége 15, 513–523 (1946).

The methods of a previous paper [Papy, same Bull. 15, 77–83 (1946); these Rev. 9, 76] are here applied to the classification of certain alternating cubic forms (or 3-indexed skew-symmetric tensors) σ, under that subgroup of the linear group leaving the quadratic form $\Gamma = 11'+22'+33'$ invariant ($11'$ is the alternating product of vectors x_1, y_1; etc.). The authors consider those forms which are in involution with Γ, i.e., for which the alternating product $[\sigma\Gamma]=0$. Under these conditions, they find, for rank not exceeding 6, the following forms:

$$\begin{array}{ll} 123, & 3(11'-22')+\lambda 3'(12+1'2'), \\ 3(12\pm 1'2'), & 3(12-1'2')+\lambda 3'(12'-21'), \\ 123+\lambda 1'2'3', & 1'23+12'3\pm 123', \end{array}$$

where λ is a canonical scalar invariant. The $\pm$ signs can all be made $+$ in the complex domain.

In both papers, the authors are aware of the known classification under the general linear group only up to rank 6 in the real, and 7 in the complex. However, the forms in the complex for rank 8 are given by G. Gurevič [C. R. (Doklady) Acad. Sci. URSS (N.S.) 7 (1935 II), 353–356]. The forms in the real of rank 7 and 8 have also been found [dissertation of the reviewer, Massachusetts Institute of Technology, 1940]. *L. C. Hutchinson* (Brooklyn, N. Y.).

Abstract Algebra

Baer, Reinhold. Direct decompositions into infinitely many summands. Trans. Amer. Math. Soc. 64, 519–551 (1948).

Verf. beweist Verfeinerungssätze für unendliche Zerlegungen der Elemente von vollständigen, modularen Verbänden unter den zusätzlichen Voraussetzungen: wenn $u<v\leqq\sum_{s\in S}s$, dann existieren Elemente $s_1, \cdots, s_n\in S$ mit $u(s_1+\cdots+s_n)<v(s_1+\cdots+s_n)$; es existiert ein Element 0 mit $0\leqq x$ für alle x. Er macht die Definitionen: gilt $a=b+c$ und $0=bc$, so heisst a "direkte" Summe von b und c: $a=b\oplus c$; die Summe aller Elemente $e(d+a)(d+b)$ mit $x=d\oplus e=a\oplus b$ heisst "Zentrum" $z(x)$ von x; gilt für alle direkten Summanden d von p: wenn $d=a\oplus b=e\oplus f$, dann existieren Elemente $a', \cdots, f''$ mit $a=a'\oplus a''$, $b=b'\oplus b''$, $e=e'\oplus e''$, $f=f'\oplus f''$, $a'\oplus b'=b'\oplus e'=e'\oplus f'=f'\oplus a'$, $a''\oplus b''=b''\oplus f''=f''\oplus e''=e''\oplus a'$, so heisst p "verfeinerbar." Verf. u.a. beweist die Sätzen: zwei direkte Zerlegungen einer verfeinerbaren Elementes in unzerlegbare Elemente sind isomorph; wenn p verfeinerbar ist und für $z(p)$ der auf- oder absteigende Kettensatz gilt, dann existieren für alle Paare von direkten Zerlegungen isomorphe Verfeinerungen. Bezüglich der vielen Nebenergebnisse und der geeigneten Definition von "isomorph" sei auf eine früheren Arbeit des Verf. [Trans. Amer. Math. Soc. 62, 62–98 (1947); diese Rev. 9, 134] und auf die Arbeit selbst verwiesen. *P. Lorenzen* (Bonn).

Nöbeling, Georg. Topologie der Vereine und Verbände. Arch. Math. 1, 154–159 (1948).

Following O. Ore [Duke Math. J. 10, 761–785 (1943); these Rev. 5, 170], the author considers lattices L which have a closure operation satisfying $\bar{A}\geqq A$, $\bar{\bar{A}}=\bar{A}$, and $A\geqq B$ implies $\bar{A}\geqq\bar{B}$. He generalizes the usual topological definitions of continuous mappings, separation axioms, compactness, etc., to such nondistributive "closure algebras" [cf. J. C. C. McKinsey and A. Tarski, Ann. of Math. (2) 47, 122–162 (1946); these Rev. 7, 359]. All results are announced; the proofs will be given elsewhere. Perhaps the most interesting result is as follows. In any L, for any directed set of elements A_α, define

$$\overline{\lim\text{top}}\,A_\alpha = \bigwedge_\alpha \Big(\overline{\bigvee_{\lambda>\alpha} A_\lambda}\Big),$$

$$\underline{\lim\text{top}}\,A_\alpha = \bigwedge_\sigma \Big(\overline{\bigvee_{\tau>\sigma} A_{\lambda(\tau)}}\Big),$$

where $\lambda(\sigma)$ denotes the most general cofinal subsets of $\{A_\alpha\}$. Clearly $\overline{\lim\text{top}}\,A_\alpha\geqq\underline{\lim\text{top}}\,A_\alpha$; in the case both are equal to the same element A, say $\lim\text{top}\,A_\alpha=A$. The author asserts that if L is complete and has a countable basis, then L is sequentially compact (every countable sequence contains a topologically convergent subsequence). *G. Birkhoff* (Cambridge, Mass.).

Specht, W. Die linearen Beziehungen zwischen höheren Kommutatoren. Math. Z. 51, 367–376 (1948).

The commutator of two elements x_1, x_2 in a ring R is defined as $k_2(x_1, x_2)=x_1x_2-x_2x_1$, and the higher commutator $k_n(x_1, \cdots, x_n)$ is defined by induction on n as yx_n-x_ny, for $y=k_{n-1}(x_1, \cdots, x_{n-1})$. It can be regarded as a form in the n indeterminates $x_1, \cdots, x_n$ over a field F of characteristic 0; any such form is an expression $f=\sum_P f_P x^P$, summed over the permutations P of the symmetric group S_n, and with a_P in F and $x^P=x_{P(1)}\cdots x_{P(n)}$. Each Q in S_n operates on the forms f as $f^Q=\sum_P f_P x^{PQ}$. By means of an application of representation theory, it follows that any submodule of the S_n-module of all forms is generated by a form f corresponding to an idempotent element $\sum_P f_P x^{PQ}$ of the associated group ring. Theorem: if the nth commutator k_n has the expression $k_n=\sum_P c_P x^P$, then the various nth commutators obtained by permutation of the arguments of k_n satisfy the linear relation $nk_n=\sum_P c_P k_n{}^P$, and all linear relations between these commutators can be obtained from this relation by linear combinations and applications of $Q\epsilon S_n$. The proof of this theorem (which was known for the cases 2 and 3) uses also suitable explicit computations in the group ring. *S. MacLane* (Chicago, Ill.).

Nakayama, Tadasi. Semilinear normal basis for quasifields. Amer. J. Math. 71, 241–248 (1949).

Let Ω be a quasifield, $\mathfrak{G}=\{G_1, G_2, \cdots, G_g\}$ a finite group of its automorphisms, Φ a subquasifield of Ω invariant as a whole under $\mathfrak{G}$, and $\mathfrak{H}$ the subgroup of $\mathfrak{G}$ which leaves Φ elementwise invariant. Under the assumption (A) that the groups $\mathfrak{G}/\mathfrak{H}$ and $\mathfrak{H}$ of automorphisms of Φ and Ω, respectively, are outer, the author proves the following result. If $(\Omega:\Phi)_r=q(\mathfrak{G}:1)+v$, where $0\leqq v<(\mathfrak{G}:1)$, there exist q elements ξ_i ($i=1, 2, \cdots, q$) in Ω and an element η such that $\xi_i^{G_j}$ ($i=1, 2, \cdots, q$; $j=1, 2, \cdots, g$) and suitable v among η^{G_j} ($j=1, 2, \cdots, g$) together form a linearly independent right-basis of Ω over Φ. In an addendum the author states that assumption (A) may be replaced by the condition that $\mathfrak{G}$ is outer. Riblet's theorem of differential basis [same J. 63, 347–351 (1941); these Rev. 2, 346] is extended to quasifields. *B. N. Moyls* (Vancouver, B. C.).

Zelinsky, Daniel. Topological characterization of fields with valuations. Bull. Amer. Math. Soc. 54, 1145–1150 (1948).

S'appuyant sur le résultat antérieur de Kaplansky [Duke Math. J. 14, 527–541 (1947); ces Rev. 9, 172] caractérisant les corps topologiques dont la topologie peut être définie par une valeur absolue (réelle), l'auteur démontre que pour que la topologie d'un corps commutatif K puisse être définie par une valuation (à valeurs dans un groupe totalement ordonné quelconque), il faut et il suffit que: (1) il existe un sous-groupe additif G de K qui est ouvert et borné (c'est-à-dire pour tout voisinage V de 0, il existe un voisinage U de 0 tel que $GU\subset V$); (2) pour tout ensemble $A\subset K$ auquel 0 n'est pas adhérent, A^{-1} est borné.

J. Dieudonné (Nancy).

Hasse, Helmut. Existenz und Mannigfaltigkeit abelscher Algebren mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. I. Math. Nachr. 1, 40–61 (1948).

Let Ω be a given field and let $\mathfrak{g}$ be a group of finite order consisting of automorphisms of Ω. An extension $\mathfrak{G}$ of an Abelian group $\mathfrak{A}$ of finite order with the factor group $\mathfrak{g}$, $\mathfrak{G}/\mathfrak{A}\cong\mathfrak{g}$, can be defined by means of a homomorphic mapping $s\rightarrow\sigma(s)$ of $\mathfrak{g}$ into the group of automorphisms of $\mathfrak{A}$ and of a factor set $C_{s,t}$ in $\mathfrak{A}$; $s, t\epsilon\mathfrak{g}$. The paper is concerned with the construction of all fields K, Abelian with the Galois group $\mathfrak{A}$ over Ω, such that K is a Galois field with the group $\mathfrak{G}$ over the invariant field Ω_0 of $\mathfrak{g}$ in Ω. It appears that it is best to generalize the problem by replacing the concept of a Galois field by that of a Galois algebra. Here, a Galois algebra K/Ω_0 with the group $\mathfrak{G}$ is defined as a semisimple commutative algebra K over Ω_0 which satisfies the following two conditions. (I) The group $\mathfrak{G}$ possesses an isomorphic representation by automorphisms of K which leave every element of Ω_0 fixed. (II) If K is considered as an Ω_0-left module with the elements of $\mathfrak{G}$ as right operators, then K is the representation module of the regular representation of $\mathfrak{G}$. The condition (II) is equivalent to the existence of a normal basis of K/Ω_0 consisting of an element θ of K and its conjugates θ^s, $s\epsilon\mathfrak{G}$. The Galois algebra K is a direct sum of fields which are all isomorphic over Ω_0. If K_0 denotes one of these fields, the Galois group of K_0/Ω_0 is a subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$. Conversely, every field K_0 of this type gives rise to a unique Galois algebra K/Ω_0 with the group $\mathfrak{G}$.

We now replace the extension problem for fields as formulated above by that of finding all Galois algebras K/Ω with the Abelian group $\mathfrak{A}$, such that K is a Galois algebra with the group $\mathfrak{G}$ over Ω_0. This really means that we study the extension problems for fields for all subgroups $\mathfrak{A}_0$ of $\mathfrak{A}$ and the corresponding subgroups $\mathfrak{G}_0$ of $\mathfrak{G}$. It is assumed that the characteristic of Ω does not divide the maximal order n of elements of $\mathfrak{A}$ and that Ω contains the nth roots of unity. A certain set of equations is formed by means of the quantities $\chi(C_{s,t})$, where χ is a character of $\mathfrak{A}$. It is necessary for the existence of algebras K that these equations have a solution in Ω. Let $\mathfrak{g}_\chi$ denote the set of all elements s of $\mathfrak{g}$ such that $\chi(\alpha)^s=\chi(\alpha^{\sigma(s)})$ for $\alpha\epsilon\mathfrak{A}$ and let Ω_χ denote the invariant field of $\mathfrak{g}_\chi$ in Ω. Form the crossed product A_χ of Ω/Ω_χ and its Galois group $\mathfrak{g}_\chi$ with the factor set $\chi(C_{s,t})$, $s, t\epsilon\mathfrak{g}_\chi$. It is shown that the system of equations mentioned above has a solution if and only if the algebras A_χ/Ω_χ split completely for every χ. The equations then have essentially only one solution. The construction of all Galois algebras K satisfying the required conditions depends on the solution of a second system of equations. The author makes the conjecture that this second system can always be solved if the first system has a solution.

The work is related to an investigation of the reviewer [J. Reine Angew. Math. 168, 44–64 (1932); Ann. of Math. (2) 48, 79–90 (1947); these Rev. 8, 310] in which, however, the group $\mathfrak{A}$ is assumed to be cyclic. [In this connection, the paper by K. Shoda, Jap. J. Math. 11, 21–30 (1934), should be mentioned.] *R. Brauer* (Ann Arbor, Mich.).

Jacobson, Nathan. Lie and Jordan triple systems. Amer. J. Math. 71, 149–170 (1949).

Dans une algèbre associative A, on pose $[a, b]=ab-ba$, $\{a, b\}=ab+ba$. Un système triple de Lie (resp., Jordan) est un sous-espace stable pour $[[a, b], c]$ (resp., $\{\{a, b\}, c\}$). L'auteur prouve de nombreuses identités liées à celle de Jacobi et aux algèbres de Jordan, puis définit l'algèbre de Lie universelle L d'un système triple de Lie L_3: c'est une algèbre de Lie engendrée par un système isomorphe à L_3, et telle que tout homomorphisme de L_3 sur un système triple de Lie L_3' soit prolongeable en un homomorphisme de L sur l'algèbre de Lie enveloppante de L_3'. On définit de même l'algèbre associative universelle de L_3.

L'auteur étudie ensuite les systèmes triples de Lie de la théorie du meson; ils possèdent une base x_i $(1\leqq i\leqq n)$ et sont définis par les relations: (1) $[[x_i, x_j], x_k]=\delta_{ki}x_j-\delta_{kj}x_i$; (2) $\phi(x_1)=0$, où ϕ est un polynôme à priori indéterminé. L'algèbre de Lie universelle de (1) est $\mathfrak{S}_{n+1}$ (matrices antisymétriques de rang $n+1$); il s'ensuit que l'algèbre associative universelle du système (1), (2) est sans radical, ses composantes irréductibles correspondant biunivoquement aux représentations irréductibles de $\mathfrak{S}_{n+1}$ dans lesquelles (2) a lieu. Les résultats connus de É. Cartan et H. Weyl permettent alors de déterminer les polynômes ϕ et les algèbres universelles correspondantes; si S désigne la représentation fidèle de $\mathfrak{S}_{n+1}$ associée à l'algèbre de Clifford de dimension 2^n, les puissances S^p de S donnent naissance à des représentations fidèles des algèbres en question. *R. Godement.*

Gotô, Morikuni. On algebraic Lie algebras. J. Math. Soc. Japan 1, 29–45 (1948).

A Lie algebra L of matrices is called algebraic if every replica of any element of L is still in L [cf. Chevalley and Tuan, Proc. Nat. Acad. Sci. U. S. A. 31, 195–196 (1945); these Rev. 7, 4; for the notion of replica of a matrix, cf. Chevalley, Amer. J. Math. 65, 521–531 (1943); these Rev. 5, 171]. The author proves completely a certain number of results whose proofs were only outlined in the note by H. F. Tuan and the reviewer. Moreover, the notion of algebraicity is generalized to abstract Lie algebras; such an algebra is called algebraic if its linear adjoint algebra is algebraic in the sense defined above, i.e., if it is l-algebraic (in the terminology of the author). The main result of the paper is that any abstract algebraic Lie algebra is isomorphic with some l-algebraic algebra of matrices. *C. Chevalley.*

Matsushima, Yozô. On algebraic Lie groups and algebras. J. Math. Soc. Japan 1, 46–57 (1948).

The reviewer and H. F. Tuan have outlined the proof of the fact that the l-algebraic Lie algebras over the field of complex numbers are the Lie algebras of the algebraic groups of matrices [for terminology and references, cf. the preceding review]. This theorem is proved completely in the present paper, by a different method. Moreover the result is generalized to the case of the abstract algebraic Lie

algebras [cf. the preceding review] over the field of complex numbers: they are the Lie algebras of the complex algebraic Lie groups (groups in which there exists a coordinate system such that the coordinates of the product of two elements are algebraic functions of the coordinates of the factors). Another theorem contained in this paper is to the effect that a Lie algebra of matrices which has a base $\{X_1, \cdots, X_n\}$ such that every replica of every X_i in the algebra is l-algebraic. *C. Chevalley* (Paris).

Weyl, Hermann. Elementary algebraic treatment of the quantum mechanical symmetry problem. Canadian J. Math. 1, 57–68 (1949).

A basic result in the mathematical analysis of atomic spectra is established here by direct and elementary methods. Let i be a "positional" variable ranging over $1, \cdots, n$ and ρ a spin variable ranging over $1, \cdots, \nu$; let Ω be the (linear) space of all anti-symmetric functions of f pairs $(i_1, \rho_1), \cdots, (i_f, \rho_f)$, Σ the space of all functions of $i_1, \cdots, i_f$, and P the space of all functions of $\rho_1, \cdots, \rho_f$. A symmetric linear transformation (i.e., one which commutes with the operators induced by permutations) A (or B) on Σ (or P) then induces a linear transformation A^* (or B^*) on Ω, and the set of all such A^* (or B^*) constitutes an algebra $\mathfrak{A}^*$ (or $\mathfrak{B}^*$). The main result is that, for $\nu=2$, $\mathfrak{A}^*$ can be completely decomposed into a direct sum of full matrix algebras $\mathfrak{A}_u^*$, where the index u ("valence defect") is an integer satisfying the restrictions $u \geqq 0$, $n-f+u \geqq 0$, w ("valence") $=2u-f \geqq 0$; $\mathfrak{A}_u^*$ occurs with multiplicity w, and the degree of $\mathfrak{A}_u^*$ is given by an explicit formula. ("Completeness" signifies that the sums of the squares of these degrees is the dimension of $\mathfrak{A}^*$.)

The proof is self-contained, except for the use of the Clebsch-Gordan formula, in obtaining the decomposition of $\mathfrak{A}^*$. In this respect it differs from the author's earlier treatment [Gruppentheorie und Quantenmechanik, 2d ed., Hirzel, Leipzig, 1931, chapter V] of a more general situation, in which Young's symmetry operators are utilized. In the course of showing completeness the following results are obtained: $\mathfrak{A}^*$ is the set of all linear transformations on Ω which commute with each element of $\mathfrak{B}^*$ (for arbitrary ν); the only linear transformations on Σ which commute with all symmetric transformations are the linear combinations of operators induced by permutations. *I. E. Segal.*

THEORY OF GROUPS

★Ledermann, Walter. Introduction to the Theory of Finite Groups. Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, N. Y., 1949. viii+152 pp. 8/6, Great Britain; $2.65, U. S. A.

This book is intended as an introduction to the theory of finite groups for undergraduate students specializing in mathematics. With this end in view, the author, in his own words, has "never hesitated to sacrifice completeness for breadth or to reject more modern methods when he considered alternative presentations to be more intelligible." The result is a very lucid introduction to the subject which would make a suitable text for a one-semester course. The following paraphrase of the table of contents will summarize the subject matter treated. Chapter I discusses the group postulates, a variety of examples, the multiplication table, isomorphisms, and cyclic groups; Chapter II, multiplication of complexes, coset decompositions, Lagrange's theorem, etc.; Chapter III, permutation groups, including transitivity and primitivity, and the groups of the regular solids and polygons; Chapter IV, the theory of normal subgroups, quotient groups, homomorphisms, the first and second laws of isomorphism, the Jordan-Hölder theorem, and the simplicity of the alternating group. Chapter V contains the Sylow theorems and a few properties of p-groups and finally Chapter VI deals with the basis theorem and invariants of Abelian groups.

The exposition is clear with frequent references to instructive examples. The common special groups are discussed in some detail including the determination of all groups of order less than or equal to eight. Problem sets occur at the end of each chapter and answers and hints on the more difficult problems are given. The typography is good and misprints are few. The reviewer could only find two: p. 110, l. 26 for G read A; p. 137, l. 11 omit commas. The bibliography is adequate but incomplete since reference is made to the second edition [1927] of Speiser's Theorie der Gruppen von endlicher Ordnung rather than to the third [1937], and no reference is made to Zassenhaus's Lehrbuch der Gruppentheorie [1937].

The major criticism of the book which the reviewer feels called upon to make is the very minor place accorded to the idea of a homomorphic mapping. This seems a pity in view of the importance of this concept in all branches of abstract algebra. In the reviewer's opinion the expositions of chapter IV, especially, would be simplified even for undergraduate students if the author's exploitation of the calculus of complexes were replaced by a systematic discussion of homomorphisms. The student would then have less difficulty in making the transition to infinite groups and to other branches of algebra. Another, but much less important, example of lack of liaison with other algebraic disciplines is the use of the terms H.C.F. for the crosscut, and L.C.M. for the union of two subgroups. While from one point of view this notation is justified, it has to be reversed when dealing with crosscuts and unions of ideals. As a final criticism of terminology the reviewer questions the suitability of using the word automorphism to mean a one-to-one mapping of an arbitrary set into itself.

D. C. Murdoch (Vancouver, B. C.).

Dias Agudo, F. A theorem about the lattice of subgroups of a group. Gaz. Mat., Lisboa 9, no. 37–38, 18–19 (1948). (Portuguese)

The lattice of subgroups of the octic group is not modular: the lattice of subgroups of any group of size less than 8 is modular. A detailed proof is given. *H. A. Thurston.*

Springer, T. A. On induced group characters. Nederl. Akad. Wetensch., Proc. 51, 699–707 = Indagationes Math. 10, 250–258 (1948).

It was proved by the reviewer [Ann. of Math. (2) 48, 502–514 (1947); these Rev. 8, 503] that every character of a group G of finite order is a linear combination with integral rational coefficients of characters of G induced by linear characters of subgroups. The author gives some simplifications of the proof. [Simplifications of the proof have also been given by the reviewer in his Colloquium Lectures.] The following result of the author may be men-

tioned. Let χ be a character of G and let H be a subgroup of G. There exists a linear combination with rational coefficients of characters of G induced by characters of H such that this linear combination has the same value as χ for all classes of conjugate elements of G containing elements of H. It is clear that the linear combination will vanish for all classes of G which do not contain elements of H.

R. Brauer (Ann Arbor, Mich.).

Neumann, B. H. On ordered groups. Amer. J. Math. **71**, 1–18 (1949).

The author proves various results about (simply) ordered groups. If $[x, y]$ denotes $x^{-1}y^{-1}xy$, then $[[x^m, y], x^n]=1$ for some $m\neq 0$, $n\neq 0$ imply $[x, y, x]=1$, whereas $[[x^2, y], y]=1$ need not imply $[[x, y], y]=1$. Let G be any abstract group; define $H_0=G$, $H_{i+1}=[G, H_i]$, and so on transfinitely. The author proves that if $\wedge H_i=1$, and if H_i/H_{i+1} has no elements of finite order except 1, then G can be made into an ordered group. (A still more general result is given.) He exhibits an ordered group which coincides with its commutator subgroup.

G. Birkhoff (Cambridge, Mass.).

Iwasawa, Kenkichi. On linearly ordered groups. J. Math. Soc. Japan **1**, 1–9 (1948).

Let G be any abstract group and let $G=F/N$ be any representation of G as a quotient-group of a free group F. The author shows that if G can be made into an ordered group, then there exists a simple ordering of F, under which N is an l-ideal and $G=F/N$ is an ordered group. He remarks that the intrinsic topology of any free group defined by the order relations introduced above always coincides with that originally defined by the reviewer [Ann. of Math. (2) **38**, 39–56 (1937), p. 53]. Finally, he gives necessary and sufficient conditions that an abstract group be isomorphic with an ordered group.

G. Birkhoff (Cambridge, Mass.).

Livšic, A. H. On the theory of direct decompositions of groups. Doklady Akad. Nauk SSSR (N.S.) **64**, 289–292 (1949). (Russian)

The author extends a result due to Kuroš [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 47–72 (1946); these Rev. **8**, 309]. An L-subgroup of a group G is defined to be a subgroup of the center Z of G which is a homomorphic image of G/K, where K is the commutator of G. The author proves the theorem that if each L-subgroup of G is periodic (with a suitable minimal condition on its primary components) then each two direct decompositions of G have centrally isomorphic refinements. Use is made of the work of Baer on direct decompositions [Trans. Amer. Math. Soc. **62**, 62–98 (1947); Bull. Amer. Math. Soc. **54**, 167–174 (1948); these Rev. **9**, 134, 410]. The proof proceeds from the case due to Kuroš, where the number of summands in each decomposition is finite, by intermediate stages to the case where the summands in one or both of the decompositions are indexed by arbitrary classes. The theorem is proved by means of a lemma which employs the concept of p-F-group. A group G is a p-F-group if it has a single subgroup which is unique with the property of being primary, belonging to the prime p, and being an L-subgroup of G. [On page 289, the equations $A'\times B'=B'\times D'=D'\times E'=E'\times A'$ should be replaced by $A'\times B'=B'\times C'=C'\times D'=D'\times A'$.]

F. Haimo (St. Louis, Mo.).

Kaplan, Samuel. A zero-dimensional topological group with a one-dimensional factor group. Bull. Amer. Math. Soc. **54**, 964–968 (1948).

Let $\{\lambda\}$ be a set of indices of cardinality c, and for each λ let R_λ be a copy of the real line, considered as a topological group in its usual topology. Let R be the weak product of all these R_λ, topologized through the norm $\|r\|=\sum|r_\lambda|$. It is first shown that R is zero-dimensional. Let R_* be the real numbers with the discrete topology, $G=R_*\times R$, and topologize G through the norm $\|(r_*, r)\|=\|r_*\|+\|r\|$ so G is also a zero-dimensional topological group. Map the indices λ one-to-one onto the interval $\frac{1}{2}\leqq c<1$, λ corresponding to c_λ. If H is the kernel of the continuous homomorphism of G onto the reals given by $(r_*, r)\rightarrow r_*+\sum c_\lambda r_\lambda$, then the topology induced by G on G/H is shown to be the usual topology of the real numbers, hence G/H is one-dimensional.

W. Ambrose (Cambridge, Mass.).

Yen, Chih-ta. Sur les polynomes de Poincaré des groupes simples exceptionnels. C. R. Acad. Sci. Paris **228**, 628–630 (1949).

The polynomials exhibited in this note are asserted to be the Poincaré polynomials of the exceptional simple compact groups. The proof is sketched in barest outline and is based on a result of Hirsch, communicated to the author, expressing the Poincaré polynomials of a homogeneous space G/g in terms of those of G and g.

P. A. Smith.

Kaplansky, Irving. Groups with representations of bounded degree. Canadian J. Math. **1**, 105–112 (1949).

A group G is said to satisfy condition P_n if for every n elements in G the set of $n!/2$ products obtained from even permutations coincides with the set of $n!/2$ products obtained from odd permutations. (Note that P_2, and also P_3, are equivalent to Abelianness.) For the special case of a compact group G the main theorem of this paper reduces to: all irreducible representations have degree less than or equal to n if and only if G satisfies condition P_r, where r is a certain function of n. The general statement of the main theorem is the same except that compact is replaced by unimodular and the set of all irreducible representations is replaced by the set of all primitive representations (a primitive representation is one obtained in the natural way from a maximal ideal in the L_1-algebra of G). It is shown that all primitive representations are of bounded degree if and only if the L_1-algebra satisfies a polynomial identity. Several other theorems relating to existence, connectedness and Abelianness of groups satisfying P_n are given. Some results of Halmos on automorphisms of compact Abelian groups are generalized to the non-Abelian case.

W. Ambrose (Cambridge, Mass.).

Kaplansky, Irving. Primary ideals in group algebras. Proc. Nat. Acad. Sci. U. S. A. **35**, 133–136 (1949).

It is shown that if G is the direct product of a compact group by a locally compact Abelian group then every closed primary ideal in the L_1-algebra of G is maximal. This theorem was previously known for the case where G is the reals or the integers. The proof makes use of the structure theorems for Abelian groups. Several applications are given, including a theorem that all irreducible representations of such a G are finite dimensional. A theorem is also given which asserts that under certain conditions a closed ideal is the intersection of maximal ideals.

W. Ambrose.

Riss, Jean. La dérivation dans les groupes abéliens localement compacts. C. R. Acad. Sci. Paris **227**, 664–666 (1948).

Riss, Jean. Les distributions dans les groupes abéliens localement compacts. C. R. Acad. Sci. Paris **227**, 809–810 (1948).

Let G be a locally compact Abelian topological group and R the group of real numbers. It is proved that the set of elements of G which lie in the image of some continuous homomorphism from R is dense in the component of the identity. If r is a continuous homomorphism from R into G and f is a (complex-valued) function on G then f is said to have an r-derivative if for each x in G the t-function $f(x+r(t))$ has a derivative at $t=0$. If f has an r-derivative for each r it is called differentiable. Let Q be the class of infinitely differentiable functions, with continuous derivatives, and of compact support; let Q_E be the subclass of those which vanish outside E. It is shown that Q is dense, in the compact-open topology, in the set of all continuous functions. Now consider on Q_E the topology in which convergence of a family of functions means uniform convergence of the functions and each of their derivatives on every compact set in E. Following L. Schwartz a distribution is defined to be a continuous linear function on Q_E. It is shown that if $\{O_\lambda\}$ is an open conditionally compact locally finite covering of E, $\{T_\lambda\}$ is a family of distributions, T_λ being defined on O_λ, and if T_λ and T_μ coincide wherever both are defined, then there is a unique distribution on Q_E which extends all the T_λ. A distribution T is called a measure derivative if it has the form $Tf=\int Dfd\mu$, where μ is a measure and D is a derivative on G. It is proved that in Q_C, for C compact, every distribution is a linear combination of measure derivatives. Let Q^* be defined like Q, but without the restriction that the functions have compact support. Every g in Q^* defines a distribution by $g(f)=\int gfdx$. It is proved that distributions of this type are dense, in a certain topology, in the set of all distributions. This makes possible a definition of the derivative of a distribution.

W. Ambrose (Cambridge, Mass.).

Riss, Jean. Sur la dérivation dans les groupes abéliens localement compacts. C. R. Acad. Sci. Paris **227**, 1194–1195 (1948).

Let G be a locally compact Abelian topological group and O the class of all continuous functions on G with compact support, and which are indefinitely differentiable and with continuous derivatives. Here a derivative of a function $f(x)$ on G is defined, for any continuous homomorphism $r(t)$ of R (the group of real numbers) into G, as the function d_rf whose value at x is $(d/dt)f(x+r(t))|_{t=0}$. A function is called differentiable if it has a derivative for every such $r(t)$. The author defines $R(G)$ to be the vector space of all continuous homomorphisms r of R into G and topologizes it in a natural way. For each $f\epsilon O$, he defines R_f to be the subspace of $R(G)$ formed by those r with $d_rf=0$ and H_f to be the closure in G of all $r(t)$ with $r\epsilon R_f$ and $t\epsilon R$. Theorem 1 states: $R(G)$ is isomorphic to a product of real lines, R_f is closed in $R(G)$, $R(G)/R_f$ is finite dimensional, f is constant on the cosets modulo H_f, f when considered as a function on G/H_f is indefinitely differentiable with continuous derivatives, and $R(G/H_f)$ is isomorphic to $R(G)/R_f$. Theorem 2 states that if T is a distribution (i.e., a continuous linear function on O) with compact support K then there exist a finite set of derivatives $\{D_i\}$ such that $D_if=0$ on K for all i implies $Tf=0$.

W. Ambrose (Cambridge, Mass.).

Ambrose, W. The L_2-system of a unimodular group. I. Trans. Amer. Math. Soc. **65**, 27–48 (1949).

L'auteur définit un H-système comme étant un espace de Hilbert muni d'une multiplication (en général non partout définie) et d'une involution; exemple: l'espace L^2 d'un groupe localement compact unimodulaire. Tout élément hermitien d'un H-système admet une décomposition spectrale au moyen d'une famille e_λ d'idempotents. Si $E\subset H$ est une famille commutative maximale d'idempotents, tout idéal bilatère fermé $I\subset H$ est engendré par $I\cap E$. L'auteur étudie ensuite, au moyen de l'algèbre de Boole des idempotents, la structure des H-systèmes commutatifs; enfin, il expose une décomposition en somme continue d'un H-système rattaché à la théorie des groupes discrets et des facteurs approximativement finis.

Note du rapporteur. (1) Sur un groupe, toute mesure de Radon de type positif, invariante par les automorphismes intérieurs, définit un H-système; les "caractères" sont vraisemblablement associés aux H-systèmes "simples"; L^2 correspond à la mesure ϵ: masse $+1$ en e. (2) Dans L^2, les idempotents ont été étudiés par B. Levitan [Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 407–427 (1946); ces Rev. **9**, 7] et par le rapporteur [mêmes Trans. **63**, 1–84 (1948); voir aussi C. R. Acad. Sci. Paris **222**, 529–531 (1946); ces Rev. **9**, 327; **7**, 454].

R. Godement (Nancy).

Godement, Roger. Sur la transformation de Fourier dans les groupes discrets. C. R. Acad. Sci. Paris **228**, 627–628 (1949).

Let G be a discrete group and let Δ be an Abelian subgroup of G. The author sets up a natural one-to-one correspondence between the characters χ of Δ and certain unitary representations U_χ of G. This correspondence combined with the decomposition of the regular representation of Δ which is effected by the Fourier transform yields an explicit decomposition of the left regular representation of G as a direct integral of the U_χ. The U_χ are all irreducible if and only if every left coset of Δ other than Δ itself meets infinitely many right cosets of Δ. The author remarks that the hypothesis of discreteness is needed only for this last result.

G. W. Mackey (Chicago, Ill.).

Gel'fand, I. M., and Naĭmark, M. A. The analogue of Plancherel's formula for the complex unimodular group. Doklady Akad. Nauk SSSR (N.S.) **63**, 609–612 (1948). (Russian)

An analogue of Plancherel's formula is obtained for functions on the complex unimodular group G_n in n dimensions, $n\geqq 3$. The analogue for the case $n=2$ was established by the same authors in an earlier paper [Izvestiya Akad. Nauk SSSR. Ser. Math. **11**, 411–504 (1947); these Rev. **9**, 495] and the formula for $n\geqq 3$ is similar in general character to that for the case $n=2$. However, a new circumstance arises in the case $n\geqq 3$ in the existence of a new type of family among the "supplementary" irreducible representations of the group, called the "degenerate" representations ("supplementary" means that the representation is not contained in the regular representation of the group). The result is as follows, in the notation used by the authors in their determination of the representations in the "fundamental" series of G_n (an irreducible representation is in this series if it is contained in the regular representation) [Mat. Sbornik N.S. **21(63)**, 405–434 (1947); these Rev. **9**, 328]: if x is a

square integrable function on G_n, $n\geqq 3$, then

$$\int |x(g)|^2 d\mu(g)=(n!)^{-1}(2\pi)^{-(n-1)(n+2)}$$
$$\times \sum_{m_1,\cdots,m_n=-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int\left[\int |K(z',z'',\chi)|^2 d\mu(z')d\mu(z'')\right]\times a(\chi)d\rho_2\cdots d\rho_n,$$

where

$$K(z',z'',\chi)=\int x(z'^{-1}\delta\zeta z'')\beta^{-\frac{1}{2}}(\delta)\chi(\delta)d\mu(\delta)d\mu(\zeta),$$
$$a(\chi)=\prod_{1\leqq p<q\leqq n}[(n_p-n_q)^2+(\rho_p-\rho_q)^2],\qquad n_1=\rho_1=0.$$

The integral defining K is convergent (in mean) relative to the norm defined by the square root of the right side of the formula.

In case $x\epsilon L_1(G)$, then $K(z',z'',\chi)$ is the kernel of the completely continuous operator $T=\int U_{\chi,g}x(g)d\mu(g)$, regarded as an operator on functions of z', and the trace of T^*T is equal to the left side of the formula. The proof is sketched for the case $n=3$, much use being made of factorizations for elements of G_n and of a number of auxiliary functions. *I. E. Segal* (Chicago, Ill.).

Croisot, Robert. Autre généralisation de l'holomorphie dans un semi-groupe. C. R. Acad. Sci. Paris **227**, 1195–1197 (1948).

Dans une note précédente [mêmes C. R. **227**, 1134–1136 (1948); ces Rev. **10**, 353], l'auteur a étudié l'holomorphie à gauche K d'un demigroupe D. Il définit de même l'holomorphie à droite $\bar{K}$ et donne les conditions pour que $K\subseteq\bar{K}$ (K et $\bar{K}$ sont des sous groupoides du demigroupe A des applications biunivoques de D dans D). L'holomorphie H est la réunion complétée de K et $\bar{K}$; H est un semigroupe; H, K, $\bar{K}$ ont même intérieur. On étudie des relations entre les centres de K, $\bar{K}$, H. *J. Kuntzmann* (Grenoble).

Paige, Lowell J. Neofields. Duke Math. J. **16**, 39–60 (1949).

A neofield is an additive loop which has a multiplication which is distributive into the addition and under which the nonzero elements form a group. It is proved that a given group may be the group of a neofield S (whose elements are 0 and those of the group) if and only if there is a one-to-one mapping T of S onto S for which (i) $0^T=1$, (ii) $x^T\neq x$, (iii) if $0\neq b\neq 1$, there is just one x for which $x^T=xb$ and (iv) if $x\neq 0$, $(x^{-1}yx)^T=x^{-1}y^Tx$. Some necessary conditions for such mappings are given and it is proved that such mappings exist for all finite commutative groups and for all finitely-generated commutative groups which do not have a unique element of order 2. Conditions are given in terms of T that the loop should have certain properties (commutativity, the inverse properties and so on).

Points and lines may be defined as ordered 3-sets (just as for skewfields); conditions are given that in the resulting geometry the axioms of incidence should hold. If they do, the neofield is said to be planar. The loop of a finite planar neofield whose group is commutative is proved to be commutative and to have the inverse property.

Finally theorems [which follow from the author's paper, Bull. Amer. Math. Soc. **53**, 590–593 (1947); these Rev. **9**, 6] are proved about ternary rings [as defined by M. Hall, Trans. Amer. Math. Soc. **54**, 229–277 (1943); these Rev. **5**, 72] and orthogonal Latin squares; in particular, there is a Latin square orthogonal to the multiplication table of a finite commutative group G if and only if $\prod_{g\epsilon G}g=1$. *H. A. Thurston* (Bristol).

NUMBER THEORY

Nyberg, Michael. Remark on the indeterminate equation $x^2-Dy^2=\pm L$. Norsk Mat. Tidsskr. **30**, 69–71 (1948). (Norwegian)

The author shows that the two equations

$$x^2-(m^2+4)y^2=\pm m$$

are both solvable for every odd integer m. For an even m the equations are simultaneously solvable only when m is a square, $m\neq 0$. *T. Nagell* (Uppsala).

Rosenthall, E. Diophantine systems suggested by Bhascara's problem. Duke Math. J. **15**, 921–928 (1948).

By the use of two lemmas, one given by Uspensky and Heaslet [Elementary Number Theory, McGraw-Hill, New York, 1939, p. 393; these Rev. **1**, 38], and the other proved by Dickson [Introduction to the Theory of Numbers, University of Chicago Press, 1929, pp. 44–47], the author proves that all rational integers satisfying the system $x^2+y^2=z^3$, $x-y=t^2$, where $(x,y)=1$, are given without duplication by $x=r^3-3rs^2$, $y=3r^2s-s^3$, where r and s are expressed by three sets of formulas. In each set r and s are polynomials in integral parameters p, q having integral coefficients. The range of these parameters is also obtained. Similar results are likewise given for $x^3+y^3=z^2$, $(x,y)=1$, where x and y are obtained as polynomials in parameters p, q, these parameters being restricted to a precise range. Finally, by using results in an earlier paper [Amer. J. Math. **65**, 663–672 (1943); these Rev. **5**, 90] the author obtains the complete integral parametric representation of $x^3+y^3=z^2$ without the restriction $(x,y)=1$. *W. H. Gage.*

Szele, Tibor. Une généralisation de la congruence de Fermat. Mat. Tidsskr. B. **1948**, 57–59 (1948).

Gauss proved that, if x is prime, then $\sum_{d|m}\mu(d)x^{m/d}\equiv 0 \pmod m$. The author apparently did not realize that this congruence had been proved for all integral x by Kantor, Weyr, Lucas, Pellet, etc. [cf. L. E. Dickson, History of the Theory of Numbers, v. 1, Carnegie Institution of Washington, 1919, pp. 82, 84–86]. Szele gives three proofs of the general congruence; his proofs are similar to those of Dickson [1895], Thué [1910] and Grandi [1882], respectively. *P. Hartman* (Baltimore, Md.).

Gatteschi, Luigi. Una classe di polinomi irriducibili. Period. Mat. (4) **26**, 102–105 (1948).

A condition sufficient for the irreducibility (in the rational field) of the polynomial $P(x)=ax\prod_{i=1}^{n-1}(x-\alpha_i)+(x-1)^n$, where the constants are positive integers, $\alpha_i\neq\alpha_k\neq 1, 2$, is established. The condition is that a exceeds a certain quantity which depends on n and the α_i. If it is satisfied all the roots but one are shown to exceed 1. If $P(x)$ were reducible, one of its factors would have all its roots exceeding 1 which is in contradiction with $P(0)=\pm 1$. *O. Todd-Taussky.*

Gatteschi, Luigi. Costruzione di polinomi irriducibili della forma $ax\prod_{i=1}^{n-1}(x-\alpha_i)+\sum_{\nu=0}^{m} g_\nu x^{m-\nu}$. Period. Mat. (4) 26, 157–159 (1948).

By a method similar to that used earlier [cf. the preceding review] a sufficient condition is obtained for the irreducibility (in the rational field) of the polynomial $ax\prod_{i=1}^{n-1}(x-\alpha_i)+\sum_{\nu=0}^{m} g_\nu x^{m-\nu}$, $m\leqq n$, where the constants are integers, $\alpha_i\geqq 2$, $\alpha_i\neq\alpha_k$, $\sum g_\nu \alpha_i^{m-\nu}\neq 0$ and $g_m\neq 0$. The condition is that $|g_m|<\alpha_i$ and $|a|$ is sufficiently large.

O. Todd-Taussky (London).

Sierpiński, W. Remarque sur la répartition des nombres premiers. Colloquium Math. 1, 193–194 (1948).

This note contains a proof of the following theorem: to each integer n there corresponds a prime $p>n$ such that not one of the integers $p\pm j$, $j=1, 2, \cdots, n$, is a prime. The proof is simple, but uses the Dirichlet theorem about primes in an arithmetic progression. The author remarks that it would be interesting to discover an elementary proof of his result. Now that there are elementary proofs of the Dirichlet theorem his proof is technically elementary, but it would still be interesting to obtain a direct elementary proof.

R. D. James (Vancouver, B. C.).

Lehmer, D. H. A conjecture of Krishnaswami. Bull. Amer. Math. Soc. 54, 1185–1190 (1948).

Let $T(N)$ denote the number of right triangles whose perimeters do not exceed $2N$ and whose sides are relatively prime. Krishnaswami Ayyangar [Tôhoku Math. J. 27, 332–348 (1926)] conjectured that $T(N)\sim N/7$. In the present paper it is proved that $T(N)=\pi^{-2}N\log 4+O(N^{\frac{1}{2}}\log N)$.

W. H. Simons (Vancouver, B. C.).

Mirsky, L. Arithmetical pattern problems relating to divisibility by rth powers. Proc. London Math. Soc. (2) 50, 497–508 (1949).

Let $a_1, \cdots, a_l; b_1, \cdots, b_m$ be any distinct positive integers. Denote by $H(x)=H_r(x; a_1, \cdots, a_l; b_1, \cdots, b_m)$ the number of integers n such that $n+a_1, \cdots, n+a_l, n+b_1, \cdots, n+b_m$ are positive and do not exceed x, and such that the first l are r-free while the remaining m are not. A complicated asymptotic expression, for $x\rightarrow\infty$, is obtained for $H(x)$ and applied to the special case of the study of the frequency of systems of consecutive r-free numbers and of r-numbers. The orders of the error terms in the results of this paper have been reduced by the author in a later paper which has already been published [Quart. J. Math., Oxford Ser. 18, 178–182 (1947); these Rev. 9, 80].

W. H. Simons.

Mirsky, L. The number of representations of an integer as the sum of a prime and a k-free integer. Amer. Math. Monthly 56, 17–19 (1949).

The author proves that every sufficiently large integer n can be represented as the sum of a prime and a k-free integer, and gives an asymptotic formula for the number of such representations. An asymptotic expression is also stated for the number of k-free integers not exceeding n having the form $p+l$, where p is a prime, and l is a given nonzero integer.

W. H. Simons (Vancouver, B. C.).

Roth, K. F. Proof that almost all positive integers are sums of a square, a positive cube and a fourth power. J. London Math. Soc. 24, 4–13 (1949).

The author proves that almost all positive integers u are representable in the form $u=x_1^2+x_2^3+x_3^4$, where x_1, x_2 and x_3 are positive integers. The method employs the Hardy-Littlewood technique of Farey dissection and is based on some results of Davenport and Heilbronn [Proc. London Math. Soc. (2) 43, 73–104 (1937)]. A novel feature of the paper is its simplified treatment of the singular series. This is accomplished by the introduction of a function which excludes unwanted terms from the singular series.

A. L. Whiteman (Los Angeles, Calif.).

Halberstam, H. Four asymptotic formulae in the theory of numbers. J. London Math. Soc. 24, 13–21 (1949).

The author obtains asymptotic formulae, with error terms, for $\sum_{\nu=1}^{x}\phi(\nu)\phi(\nu+k)$ as $x\rightarrow\infty$, for $\sum_{\nu=1}^{n-1}\phi(\nu)\phi(n-\nu)$ as $n\rightarrow\infty$, for $\sum_{\nu=1}^{x}\sigma_r(\nu)\sigma_s(\nu+k)$ as $x\rightarrow\infty$, and for $\sum_{\nu=1}^{n-1}\sigma_r(\nu)\sigma_s(n-\nu)$ as $n\rightarrow\infty$. Here $\phi(n)$ is Euler's function, $\sigma_r(n)$ is the sum of the rth powers of the divisors of n, k is a fixed positive integer, and r and s are fixed positive numbers. The proofs, which are elementary, are similar to those used by Ingham [same J. 2, 202–208 (1927)] to obtain asymptotic formulae for $\sum_{\nu=1}^{x}d(\nu)d(\nu+k)$ and $\sum_{\nu=1}^{n-1}d(\nu)d(n-\nu)$, where $d(n)$ is the number of divisors of n.

P. T. Bateman.

Bochner, S., and Chandrasekharan, K. Summations over lattice points in κ-space. Quart. J. Math., Oxford Ser. 19, 238–248 (1948).

This paper generalizes results of Dixon and Ferrar [Quart. J. Math., Oxford Ser. 5, 48–63, 172–185 (1934)]. Let $P(n, n)$ be a positive definite quadratic form in the κ variables $n_1, \cdots, n_\kappa$; $0<\lambda_1<\lambda_2<\cdots$ are the values of $P(n, n)$ for integral vectors n; $h_1, \cdots, h_\kappa$ are real numbers; $\tau_\kappa^P(\lambda_m, h)=\sum \exp\{2\pi i(n_1h_1+\cdots+n_\kappa h_\kappa)\}$, with summation extended over all lattice points (n) which satisfy $P(n, n)=\lambda_m$. The real number ξ satisfies $\xi^2\neq Q(h+n, h+n)$ for all lattice points (n); here Q is the inverse of P. It is proved that the series

$$\sum_m \tau_\kappa^P(\lambda_m, h)\exp(2\pi i\xi\lambda_m^{\frac{1}{2}})\lambda_m^{-s}$$

is summable by Riesz's typical means of order γ if $\gamma>\frac{1}{2}(\kappa-1)$, $s>\frac{1}{2}\kappa-\frac{1}{2}-\frac{1}{2}\gamma$. The restriction $\gamma>\frac{1}{2}(\kappa-1)$ can be replaced by $\gamma\geqq 0$ if P is integer-valued. The results are obtained from results for Bessel series of the form

$$\sum_m \tau_\kappa^P(\lambda_m, h)J_\mu(2\pi\xi\lambda_m^{\frac{1}{2}})\lambda_m^{-s+\frac{1}{2}} \qquad (\mu>-1).$$

Some special cases are indicated where the sum of the latter series is zero for all admissible values of ξ.

N. G. de Bruijn (Delft).

Gloden, A. Sur la multigrade $A_1, A_2, A_3, A_4, A_5 \overset{k}{=} B_1, B_2, B_3, B_4, B_5$ $(k=1, 3, 5, 7)$. Euclides, Madrid 8, 383–384 (1948).

Announcement of two solutions each with two parameters, to be proved later. Numerical results are deduced.

N. G. W. H. Beeger (Amsterdam).

Gloden, A. Un nouveau procédé de résolution de la sextigrade normale $A_1, A_2, A_3, A_4, A_5, A_6, A_7 \overset{6}{=} B_1, B_2, B_3, B_4, B_5, B_6, B_7$. Bull. Soc. Roy. Sci. Liége 17, 252–256 (1948).

Rai, T. On a problem of additive theory of numbers. Math. Student 15 (1947), 25–28 (1948).

Let $\beta(k)$ denote the least number of kth powers whose sum is equal to the sum of a smaller number of kth powers.

Improvements of $\beta(k)$ for 4 values of k are derived. To this aim Tarry's theorem on multigrade equalities [see Dickson, History of the Theory of Numbers, v. 2, Carnegie Institution of Washington, 1920, p. 710] is applied to $2+5=3+4$ with $x=3, 5, 7, 8, 13, 11, 9, 19, 17, 25$ in succession, finally diminishing every term by 62. The result is an equality, giving $\beta(10)\leqq 11$. In an analogous manner improvements for $k=14, 15, 16$ are derived.
N. G. W. H. Beeger (Amsterdam).

Sastry, S. On equal sums of like powers. Math. Student 15 (1947), 29–32 (1948).

Tarry's theorem [see the preceding review] is applied to $0+(a+b)=a+b$ with $x=a+b$ and $x=a+2b$ in succession. Then $a=3m-n$, $b=2m-m$ is substituted. Again $a=3n-m$, $b=2m-n$. The two results give a solution of

$$\sum_{i=1}^{h} x_i^l = \sum_{i=1}^{h} y_i^l = \sum_{i=1}^{h} z_i^l, \qquad l=1, 2, 3;\ h=4.$$

Similarly a solution for $h=6$; $l=1, 2, \cdots, 5$, is derived and 3 equal sums of 3 fourth powers are found.
N. G. W. H. Beeger (Amsterdam).

Arnold, B. H., and Eves, Howard. A simple proof that, for odd $p>1$, $\arccos 1/p$ and π are incommensurable. Amer. Math. Monthly 56, 20–21 (1949).

A short proof is given for the theorem stated in the title. The authors also remark that this result is a corollary of the following well-known theorem. Let m and n ($n>2$) be two relatively prime integers; then $\cos 2\pi m/n$ is an algebraic number of degree $\varphi(n)/2$, $\varphi(n)$ denoting Euler's function.
J. Popken (Utrecht).

Butlewski, Z. A proof that e^m is irrational. Colloquium Math. 1, 197–198 (1948).

Niven [Bull. Amer. Math. Soc. 53, 509 (1947); these Rev. 9, 10] has given a simple proof for the irrationality of π. Using Niven's method the author proves that e^m is irrational for any positive integer m. The reviewer remarks that this proof resembles Hermite's proof for the same theorem [Oeuvres, v. 3, pp. 153–155]. A still more elementary proof was also given by Hermite [Oeuvres, v. 3, pp. 127–130].
J. Popken (Utrecht).

Mostowski, A. Un théorème sur les nombres $\cos 2\pi k/n$. Colloquium Math. 1, 195–196 (1948).

This is the well-known theorem that these numbers are expressible by real radicals exactly when $\varphi(n)$ is a power of 2.
G. Whaples (Bloomington, Ind.).

Popken, J. On the irrationality of π. Math. Centrum Amsterdam. Rapport ZW 1948-014, 5 pp. (1948). (Dutch)

Given two functions

$$R_0(x)=\sum_0^\infty a_{0m}x^{-2m}$$

and

$$R_1(x)=\sum_0^\infty a_{1m}x^{-2m-1},$$

with $a_{10}\neq 0$, the author recursively defines

$$R_\nu(x)=\sum_0^\infty a_{\nu m}x^{-2m-\nu}$$

by the equations

$$R_{\nu-2}(x)=\frac{a_{\nu-2,0}}{a_{\nu-1,0}}xR_{\nu-1}(x)+R_\nu(x), \qquad \nu=2, 3, \cdots.$$

Applying this algorithm to $R_0(x)=\cos x^{-1}$, $R_1(x)=\sin x^{-1}$, he proves Lambert's theorem, that $\cot y$ is irrational for rational $y\neq 0$. This implies immediately that π is irrational.

Application is also made to the problem of location of the real zeros of a polynomial $B(x)=b_0x^n+b_1x^{n-1}+\cdots$. Let D_ν be the minor determinant formed from the first ν rows and columns of the matrix

$$\begin{bmatrix} b_1 & b_0 & 0 & 0 & 0 & 0 & \cdots \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & \cdots \\ b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & \cdots \\ & \cdots & & \cdots & & \cdots & \end{bmatrix},$$

where $b_\nu=0$ for $\nu>n$. Then by applying the algorithm with slight modifications to the polynomials

$$R_0(x)=b_0x^n+b_2x^{n-2}+\cdots$$

and

$$R_1(x)=b_1x^{n-1}+b_3x^{n-3}+\cdots,$$

the author deduces Hurwitz's theorem, that if $b_0>0$ and $D_\nu>0$ for $\nu=1, 2, \cdots, n$, then $B(x)$ has no positive zeros.
W. J. LeVeque (Ann Arbor, Mich.).

Morduhaĭ-Boltovskoĭ, D. On hypertranscendental functions and hypertranscendental numbers. Doklady Akad. Nauk SSSR (N.S.) 64, 21–24 (1949). (Russian)

The author designates a function as hyperalgebraic or hypertranscendental according as it does or does not satisfy an algebraic differential equation (1) $f(x, y, y', \cdots, y^{(n)})=0$. If (1) has algebraic numbers for coefficients, a solution y of (1) is called algebraically-hyperalgebraic if y and its first $n-1$ derivatives assume algebraic values for some algebraic value of x. A value of such a function for any algebraic value of x is called a hyperalgebraic number. Such numbers are seen to form a countable set. It is shown that if a power series represents an algebraically-hyperalgebraic function, the coefficients all lie in a field generated by an algebraic number.
J. F. Ritt (New York, N. Y.).

Wintner, Aurel. On restricted partitions with a basis of uniqueness. Revista Unión Mat. Argentina 13, 99–105 (1948).

A set S of distinct positive integers will be called a π-set if it consists of all integers representable as sums of distinct elements of a sequence (K): $k_0<k_1<k_2<\cdots$ of positive integers having the property that $k_1'+\cdots+k_a'=k_1''+\cdots+k_b''$ ($k_1'<k_2'<\cdots$, $k_1''<k_2''<\cdots$ being elements of (K)) implies $a=b$, $k_1'=k_1''$, $\cdots$, $k_a'=k_b''$. If S is any set of positive integers, and $\nu(n)$ is the number of those elements of S which do not exceed n, then S will be said to be measurable if $\nu(n)/n$ tends to a limit for $n\to\infty$. The limit will be called. the measure of S. The main result of the paper is that if (K) is a basis such that $\lim_{n\to\infty} 2^n/k_n$ exists and equals λ, then the π-set generated by (K) is measurable, and its measure is precisely λ.
R. Salem (Cambridge, Mass.).

Chowla, S. On difference sets. Proc. Nat. Acad. Sci. U. S. A. 35, 92–94 (1949).

A set of residues $a_1, \cdots, a_m$ (mod g) is called a difference set if every difference $d\not\equiv 0$ (mod g) arises the same number r of times as a difference of two a's. Here $m(m-1)=r(g-1)$. The chief theorem given here states that if g contains a

prime factor $p\equiv 3 \pmod 4$ such that $-p$ is a quadratic nonresidue of some prime q dividing $m-r$ to an odd power, then no difference set of m residues modulo g exists. When $r=1$ the set is called a perfect difference set. It has been shown by the reviewer [Duke Math. J. **14**, 1079–1090 (1947); these Rev. **9**, 370] that the construction of a perfect difference set is equivalent to the construction of a cyclic projective plane. Thus the nonexistence of a perfect difference set both for $m=11$ and for 160, claimed to be new results here, are in fact special cases of theorems proved by the reviewer. *M. Hall, Jr.* (Columbus, Ohio).

Vijayaraghavan, T., and Chowla, S. On complete residue sets. Quart. J. Math., Oxford Ser. **19**, 193–199 (1948).

In the first section the authors establish the following result. If $r_1, \cdots, r_q$ and $s_1, \cdots, s_q$ are two complete residue sets $(\operatorname{mod} q)$, where $q>2$, then $r_1s_1, \cdots, r_qs_q$ is not a complete residue set $(\operatorname{mod} q)$. This is an extension of a result of Hurwitz who considered the special case where q is an odd prime. The main result of the paper is the solution of the following problem. Let n be an integer, $h=\varphi(n)$ and $r_1, \cdots, r_h$; $s_1, \cdots, s_h$ two complete primitive residue sets $(\operatorname{mod} n)$, that is, two sets of integers all prime to n and incongruent $(\operatorname{mod} n)$. Is it possible for the product set $r_1s_1, \cdots, r_hs_h$ to be a complete primitive residue set? The answer is that, if $n=2$ or if n has no primitive root, there exist suitable complete primitive residue sets $r_1, \cdots, r_h$; $s_1, \cdots, s_h$ such that $r_1s_1, \cdots, r_hs_h$ is also a complete primitive residue set. The proof of this result proceeds along the following lines. It is shown first of all that, if m and n are relatively prime and the result is true for m and n, it is true for the product mn. Secondly, the result is established for every integer n belonging to a set S of numbers of five specified forms. Finally, it follows if $n=2$ or if n has no primitive root, then either n belongs to S or can be represented as the product of numbers belonging to S. It is also pointed out that if $n>2$ and if n has a primitive root then the result is false. Thus the problem is completely solved. *R. D. James* (Vancouver, B. C.).

Vijayaraghavan, T. On the fractional parts of powers of a number. IV. J. Indian Math. Soc. (N.S.) **12**, 33–39 (1948).

[For part III cf. J. London Math. Soc. **17**, 137–138 (1942); these Rev. **5**, 35.] Let θ be a real number greater than 1. Let $G(\theta)$ denote the set of limit points of the set of fractional parts of the numbers $\theta, \theta^2, \theta^3, \cdots, \theta^n, \cdots$. The author proves, as a consequence of a more general result, that if E denotes the set of the numbers θ for which $G(\theta)$ is not the entire interval $(0, 1)$, then E has the power of the continuum. (It is known that E has Lebesgue measure zero.) He proves also that the intersection of E and any interval (a, b) $(1<a<b)$ has the power of the continuum. *R. Salem* (Cambridge, Mass.).

Koksma, J. F. On a definite integral in the theory of uniform distribution. Nieuw Arch. Wiskunde (2) **23**, 40–54 (1949).

Let $f(1), f(2), \cdots$ be an infinite real sequence. Let $\nu_{\alpha,\beta}(N)$ be the number of those among the first N elements of the sequence which belong (mod 1) to the interval $\alpha\leqq u<\beta$. Writing $\nu_{\alpha,\beta}(N)=(\beta-\alpha)N+R_{\alpha,\beta}(N)$, the term $R_{\alpha,\beta}(N)$ is called the remainder, which, in case of a uniformly distributed sequence, satisfies $R_{\alpha,\beta}(N)=o(N)$ for any α, β satisfying $\beta-\alpha\leqq 1$. The purpose of the paper is to consider sequences of the form $f(x, \theta)$ $(x=1, 2, \cdots)$ when θ is a parameter and to give the following result about the order of magnitude of $R_{\alpha,\beta}(N, \theta)$ in mean. Let $f(x, \theta)$ denote a real function which is defined for $x=1, 2, \cdots, N$ $(N\geqq 2)$, $a\leqq\theta\leqq b$, and which for fixed x has a derivative $f_\theta'(x, \theta)$ which is a positive and nondecreasing function of θ on the segment $a\leqq\theta\leqq b$. Define $L_i=L_i(N)$ $(i=1, 2, 3)$ by

$$L_1=\sum_{x=1}^{N}\frac{1}{f_\theta'(x, a)}, \quad L_2=\sum_{x=1}^{N}\frac{x}{f_\theta'(x, a)},$$

$$L_3=\sum_{x=2}^{N}\sum_{y=1}^{x-1}\int_a^b\frac{f_\theta'(y, w)}{f_\theta'(x, w)}dw.$$

Let α, β denote a pair of real numbers such that $\tau=\beta-\alpha\leqq 1$ and let $R(N)=R_{\alpha,\beta}(N;\theta)$ denote the remainder with respect to the uniform distribution (mod 1) of the numbers $f(1, \theta), f(2, \theta), \cdots, f(N, \theta)$ corresponding to the interval $\alpha\leqq u<\beta$. Then we have

$$\int_a^b R^2(N)d\theta=(b-a)\tau N-(b-a)\tau^2N \pm 16\eta\tau^2NL_1\pm 24\eta\tau L_2\pm 8\eta\tau L^3 \quad (0\leqq\eta\leqq 1).$$

Various applications are given. *R. Salem.*

Ollerenshaw, Kathleen. The critical lattices of a sphere. J. London Math. Soc. **23**, 297–299 (1948).

This paper gives an elementary proof that the determinants of admissible lattices of the unit sphere are not less than $1/\sqrt{2}$ and determines all the admissible lattices with determinant $1/\sqrt{2}$ (i.e., the critical lattices). *D. Derry* (Vancouver, B. C.).

Mikolás, Miklós. Sur l'hypothèse de Riemann. C. R. Acad. Sci. Paris **228**, 633–636 (1949).

Let ρ_ν be the νth fraction of the Farey series of order $[x]$ and let $\Phi(x)$ be the number of fractions in the series. The author shows that, if $|\lambda|<2\{5/\zeta(3)\}^{\frac{1}{2}}=4.078\cdots$ and $\lambda\neq 0$, then the Riemann hypothesis is equivalent to

$$\sum_{\nu=1}^{\Phi(x)} e^{\lambda\rho_\nu}-\frac{e^\lambda-1}{\lambda}\Phi(x)=o(x^{\frac{1}{2}+\epsilon}),$$

where ϵ is an arbitrary positive number. This is deduced from the well-known result that the Riemann hypothesis is equivalent to $\sum_{n\leq x}\mu(n)=o(x^{\frac{1}{2}+\epsilon})$. Criteria of a similar form employing the functions $\cos\lambda\rho$, $\sin\lambda\rho$, ρ^2 and ρ^3 in place of $e^{\lambda\rho}$ are stated without proof. There are numerous misprints. *R. A. Rankin* (Cambridge, England).

Davenport, Harold. Sur les corps cubiques à discriminants négatifs. C. R. Acad. Sci. Paris **228**, 883–885 (1949).

[In the original the author's first name is incorrectly given as Henri.] The following result is announced. Let K be a cubic field of discriminant $d<0$. To every integer ξ of K, there is a number λ of K such that the absolute value of the norm of $\xi-\lambda$ is always greater than $c\sqrt{(-d)}$, where c is an absolute constant. As a consequence, the Euclidean algorithm does not exist if $-d>c^{-2}$. *L. K. Hua.*

Dinghas, Alexander. Verallgemeinerung eines Hilbertschen Satzes über das Verhalten einer mit den Legendreschen Polynomen zusammenhängenden quadratischen Form. S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1948**, no. 2, 12 pp. (1948).

It is proved that if $P_n(x)$ is a polynomial of degree n with integer coefficients and $|b-a|<4$, then for sufficiently

large n these coefficients can be so chosen as to make $\int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma}P_n^2(x)dx$ ($\gamma>0, \alpha-\gamma>-1$) arbitrarily small.

H. S. A. Potter (Aberdeen).

Pall, Gordon. Composition of binary quadratic forms. Bull. Amer. Math. Soc. **54**, 1171–1175 (1948).

A compound of binary quadratic forms is defined in a manner similar to that of Dirichlet [cf. L. E. Dickson, History of the Theory of Numbers, vol. 3, Carnegie Institution of Washington, 1923, p. 66]. It is shown that, if the divisors of the classes of two forms are coprime, then their compound defines a unique class. Gauss's criterion for the equivalence of two binary quadratic forms is generalised to forms in n variables. *H. S. A. Potter* (Aberdeen).

Cassels, J. W. S. A note on the division values of $\wp(u)$. Proc. Cambridge Philos. Soc. **45**, 167–172 (1949).

Let $x=\wp(u)$, $y=\frac{1}{2}\wp'(u)$, where the Weierstrass elliptic $\wp$-function has the invariants $g_2=4A$, $g_3=4B$; then (1) $y^2=x^3-Ax-B$. [Note: the definitions of x and y are printed incorrectly in the paper and should be as above.] A classical result is that $\wp(nu)=x-\psi_{n-1}\psi_{n+1}/\psi_n^2$, where $\psi_n=\psi_n(u)=\sigma(nu)/\sigma(u)^{n^2}$, with the Weierstrass σ-function, from which it follows that $\wp(nu)$ is a rational function of $\wp(u)$. The author introduces the notation ψ_n to mean $(-1)^{n+1}$ times the classical ψ_n function. He also defines two new functions ϕ_m and ω_m by the relations: $\phi_m=x\psi_m^2-\psi_{m-1}\psi_{m+1}$, $4y\omega_m=\psi_{m+2}\psi_{m-1}^2-\psi_{m-2}\psi_{m+1}^2$. The functions ϕ_m, ψ_m^2 are polynomials in x, A, B with rational integral coefficients. The author defines congruence, $\phi_m\equiv 0 \pmod m$, to signify that all the coefficients of the polynomial ϕ_m are divisible by m. The author proves [theorem I]: $\phi_m'\equiv(\psi_m^2)'\equiv 0 \pmod m$, where $\phi_m'=(\partial/\partial x)\phi_m$, etc. If m is even, the stronger theorem II holds: let $2^k|m$ ($k>0$), then $\phi_m\equiv(x^2+A)^{m^2/2} \pmod{2^{2k+1}}$, and $2^k|\psi_m$. Theorem III is a more complicated result of the same kind for m divisible by 3.

Theorems IV and V concern the zeros of $\psi_m^2(x)$, i.e., the "division values" $x_{\lambda\mu}=\wp((\lambda\Omega_1+\mu\Omega_2)/m)$, where Ω_1, Ω_2 are a pair of basic periods and $0\leqq\lambda$, $\mu<m$. Let A, B take integral values in an algebraic number field F. A solution (x_1, y_1) of (1) ($x_1, y_1\epsilon F$) is of finite order m if and only if m is the smallest integer for which mu_1 is a period of $\wp(u)$, where u_1 is the argument corresponding to (x_1, y_1). Hence a solution of order m satisfies $\psi_l^2(x)=0$ if and only if $m|l$. Theorem IV states: if (x_1, y_1) has order p^k ($p\neq 2$), there is an integral ideal $\mathfrak{t}$ in F such that $x_1\mathfrak{t}^2$ and $y_1\mathfrak{t}^3$ are integral and $\mathfrak{t}^r|p$ ($p\neq 3$), $\mathfrak{t}^s|3$ ($p=3$), where $r=p^k-p^{k-1}$, $s=3^{2k}-3^{2k-2}$; all other solutions of finite order are integers in F. Theorem V discusses the divisibility of the discriminant $4A^3-27B^2$ with respect to y_1 and $\mathfrak{t}$. It is stated that theorems IV and V generalize results of Mahler, Lutz, Billing, Weil, and Châtelet. *J. Lehner* (Philadelphia, Pa.).

Maass, H. Über die Erweiterungsfähigkeit der Hilbertschen Modulgruppe. Math. Z. **51**, 255–261 (1948).

Let $\tau_1, \cdots, \tau_n$ be n complex variables, and let T be the domain $\mathfrak{J}(\tau_k)>0$, $k=1, \cdots, n$. If Z is a totally real algebraic number field of degree n, and $\alpha_1, \cdots, \delta_1$ are integers of Z such that $\alpha_1\delta_1-\beta_1\gamma_1=1$, Hilbert's modular group M consists of the hyperabelian transformations $\tau_k\to(\alpha_k\tau_k+\beta_k)/(\gamma_k\tau_k+\delta_k)$, where $\alpha_k, \cdots, \delta_k$ are the conjugates of $\alpha_1, \cdots, \delta_1$. It is discontinuous in T [cf. Maass, S.-B. Heidelberger Akad. Wiss. **1940**, no. 2; these Rev. **2**, 213]. The author proves that the maximal extension of M which is discontinuous in T is Hurwitz's group H, which consists of the hyperabelian transformations

$$\tau_k\to S_k(\tau_k),\quad S_k=\begin{pmatrix}\alpha_k/\sqrt{\omega_k}, & \beta_k/\sqrt{\omega_k}\\ \gamma_k/\sqrt{\omega_k}, & \delta_k/\sqrt{\omega_k}\end{pmatrix},$$

where now $\alpha_1, \cdots, \delta_1, \omega$ are integers of Z such that ω is totally positive, $\alpha_k\delta_k-\beta_k\gamma_k=\omega_k$, and the coefficients of S are algebraic integers. *R. Hull* (Lafayette, Ind.).

ANALYSIS

Matos Peixoto, Maurício. An inequality among positive numbers. Gaz. Mat., Lisboa **9**, no. 37–38, 19–20 (1948). (Portuguese)

Let $x_1, x_2, \cdots, x_n$ be nonnegative numbers of which at least two are positive and let $S=x_1+x_2+\cdots+x_n$. Then $\sum_{k=1}^n x_k/(S-x_k)\geqq n/(n-1)$. The author's proof is purely algebraic. By use of a Lagrange multiplier, one can see immediately that the left member is, under the restriction $S-x_1-x_2-\cdots-x_n=0$, a minimum when $x_1=x_2=\cdots=x_n=S/n$ and the inequality follows.

R. P. Agnew (Ithaca, N. Y.).

Kahanoff, Boris. Certaines inégalités des nombres. Bull. Inst. Égypte **29**, 323–327 (1948).

The author employs the following simple principle. Let $f(x, n)$ be a differentiable function of x for $0\leqq x<\infty$, with a single maximum $f(x_0, n)$ as a function of x. Then for any other x, $f(x, n)<f(x_0, n)$. By selecting special $f(x, n)$ and x, numerous inequalities are derived. As an example, if $f(x, n)=x^n/(1+x^{2n-2})$, $x_0=(n/(n-2))^{1/(2n-2)}$, and $x=1$, one obtains the inequality $n^n(n-2)^{n-2}>(n-1)^{2n-2}$ for $n>2$.

A. W. Goodman (Lexington, Ky.).

Jecklin, Heinrich. Ein Satz über Quadratsummen. Elemente der Math. **4**, 12–14 (1949).

The author shows that $(\binom{n}{i}-1)_i p_2\geqq 2\sum_{k=1}^i(-1)^{k-1}s_{i-k}s_{i+k}$, where s_j means the jth elementary symmetric function of $a_1, \cdots, a_n$ ($s_0=1$, $s_m=0$ for $m<0$ and $m>n$) and ${}_jp_2$ is the sum of squares of the members in s_j (e.g., for $n=3$, $s_2=a_1a_2+a_1a_3+a_2a_3$, ${}_2p_2=a_1^2a_2^2+a_1^2a_3^2+a_2^2a_3^2$). This could be proved more easily, since the root-mean-square $\{{}_ip_2/\binom{n}{i}\}^{\frac{1}{2}}$ is greater than the arithmetic mean $s_i/\binom{n}{i}$, i.e., $s_i^2\leqq\binom{n}{i}{}_ip_2$ and the proof is completed by the use of a relation, which the author needs also and which becomes evident by comparing the coefficients of x^{2i} in

$$(1+a_1x)(1+a_2x)\cdots(1+a_nx)\cdot(1-a_1x)(1-a_2x)\cdots(1-a_nx)$$
$$\equiv(1-a_1^2x^2)(1-a_2^2x^2)\cdots(1-a_n^2x^2),$$

namely: $2\sum_{k=1}^i(-1)^{k-1}s_{i-k}s_{i+k}=s_i^2-{}_ip_2\leqq[\binom{n}{i}-1]_ip_2$.

J. Aczél (Szeged).

Cassels, J. W. S. An elementary proof of some inequalities. J. London Math. Soc. **23**, 285–290 (1948).

The author proves the following inequalities:

$$(1)\qquad \int_0^1\left(\sum_{n=1}^\infty a_nx^n\right)^k dx-C(k)\cdot\sum_{n=1}^\infty a_n^k(n+1)^{k-2}\leqq 0,$$
$$k\geqq 1,\ C(\infty)=\infty$$

[G. H. Hardy and J. E. Littlewood, J. Reine Angew. Math. 157, 141–158 (1927), where also $C(1+0)=\infty$];

$$(2)\qquad A\int_0^1\left(\sum_{n=1}^{\infty}a_nx^n\right)^L dx-\sum_{n=1}^{\infty}a_n^L(n+1)^{L-2}\geqq 0,\quad 0<L\leqq 1;$$

$$(3)\qquad \sum_{m,n=0}^{k-1}\frac{a_mb_n}{(m+n+1)^L} -\left\{\int_0^{2k}\frac{dx}{x^{1/(Lp')}(1+x)}\right\}^{1/q'}\left\{\int_0^{2k}\frac{dx}{x^{1/(Lq')}(1+x)}\right\}^{1/p'} \times\left\{\sum_{m=0}^{k-1}a_m^p\right\}^{1/p}\left\{\sum_{n=0}^{k-1}b_n^q\right\}^{1/q}<0$$

[cf. Hilbert's double-series theorem: see, e.g., G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge University Press, 1934, pp. 226–259], where $p>1$, $q>1$, $p'=p/(p-1)$, $q'=q/(q-1)$, $0<1/p'+1/q'=L\leqq 1$. The proofs are based on the fact that when the left-hand sides of these inequalities, regarded as functions of the coefficients a_n (or of a_m, b_n) attain their extremal values, their derivatives are equal to zero. In (3) a direct application of Hölder's inequality, in (1) and (2) some known properties of the gamma function help to conclude the proof. The present form of (3) and its special cases are stronger than previously known results. *J. Aczél* (Szeged).

Karamata, J. Inégalités relatives aux quotients et à la différence de ∫fg et ∫f∫g. Acad. Serbe Sci. Publ. Inst. Math. 2, 131–145 (1948). (French. Serbian summary)

The author gives a direct proof [due to A. Bilimović] of the following generalization of the first mean value theorem of integral calculus, which is a special case of a previous result of the author [Acad. Roy. Serbe. Bull. Acad. Sci. Math. Nat. A. 1, 97–106 (1933)]:

$$\frac{ab(B-G)+AB(G-b)}{a(B-G)+A(G-b)}F\leqq\int_0^1 f(t)g(t)dt \leqq\frac{Ab(B-G)+AB(G-b)}{A(B-G)+a(G-b)}F,$$

where

$$0<a=\inf_{t\in(0,1)}f(t),\quad A=\sup_{t\in(0,1)}f(t),\quad b=\inf_{t\in(0,1)}g(t),\quad B=\sup_{t\in(0,1)}g(t)$$

and $F=\int_0^1 f(t)dt$, $G=\int_0^1 g(t)dt$. The proof is based on the fact that there exist such $y_1, y_2\in(a,A)$ that

$$\int_0^1 f(t)g(t)dt=bF+(G-b)y_1=BF-(B-G)y_2.$$

The author derives from this theorem and its proof the following estimations for the quotient and the difference of $\int_0^1 fg$ and $\int_0^1 f\cdot\int_0^1 g$ (in the first both a and b must be positive, in the second neither of the two):

$$\left\{\frac{(aB)^{\frac12}+(Ab)^{\frac12}}{(ab)^{\frac12}+(AB)^{\frac12}}\right\}^2\leqq\frac{\int fg}{\int f\cdot\int g}\leqq\left\{\frac{(ab)^{\frac12}+(AB)^{\frac12}}{(aB)^{\frac12}+(Ab)^{\frac12}}\right\}^2,$$

$$\left|\int fg-\int f\cdot\int g\right| \leqq\min\left\{\frac{(A-F)(F-a)}{A-a}\cdot(B-b),\ \frac{(B-G)(G-b)}{B-b}\cdot(A-a)\right\}.$$

These results are generalizations of those of P. Schweitzer Mat. Fiz. Lapok 23, 257–261 (1914)], J. Kürschák [ibid., 378 (1914)], G. Pólya and G. Szegö [Aufgaben und Lehrsätze aus der Analysis, v. 1, Springer, Berlin, 1925, p. 57] and of G. Grüss [Math. Z. 39, 215–226 (1934)] and E. Landau [Math. Z. 39, 742–744 (1935)]. The author also improves an estimate of G. Kowalewski [Z. Math. Physik 42, 153–157 (1897); 43, 118–120 (1898)]. *J. Aczél* (Szeged).

Fenchel, W. On conjugate convex functions. Canadian J. Math. 1, 73–77 (1949).

Let G be a nonnull convex set in Euclidean n-space and let f be a convex lower semicontinuous function defined on G such that $f(x)\to\infty$ as $x\to b$ for every boundary point b of G which does not lie in G. It is shown that to G and f there uniquely correspond a set Γ and function ϕ possessing the same properties as postulated for G and f such that for all x in G and all ξ in Γ the inner product inequality $x\cdot\xi\leqq f(x)+\phi(\xi)$ is satisfied, there being to each x some ξ for which the equality holds. The correspondence is reciprocal. These results concerning the general notion of conjugate convex functions f and ϕ flow simply from the following definitions: let Γ be the set of all points ξ such that the least upper bound of $x\cdot\xi-f(x)$ as x ranges over G is finite; let $\phi(\xi)$ be this least upper bound. *W. Gustin.*

Sard, Arthur. The remainder in approximations by moving averages. Bull. Amer. Math. Soc. 54, 788–792 (1948).

Many of the processes of interpolation or smoothing are of the following sort. A function $L(s)$, defined for all real s, characterizes the process. Given a function $x(s)$, the function $y(t)=\sum_{j=-\infty}^{\infty}x(j)L(t-j)$ is constructed, when possible; $y(t)$ is thought of as an approximation of $x(t)$. The remainder in the approximation is $R[x]=x(t)-y(t)$. Schoenberg has given a criterion for recognizing cases in which the approximating process is exact for polynomials of degree $n-1$; that is, cases in which $R[x]=0$, for all t, whenever $x(s)$ is a polynomial of degree $n-1$ [Quart. Appl. Math. 4, 45–49, 112–141 (1946); these Rev. 7, 487; 8, 55].

The author obtains an integral representation of such operations $R[x]$ in terms of the nth derivative $x^{(n)}(s)$ in the following way. Let $g(s,t)$ be a function which, for each number t in a given set T, is of bounded variation in s in each finite s-interval;

$$y(t)=\int_{-\infty}^{\infty}x(s)d_sg(s,t),\quad R[x]=x(t)-y(t),\qquad t\in T.$$

Assume that $R[x]$ exists and vanishes, for all $t\in T$, whenever $x(s)$ is a polynomial of degree $n-1$ ($n\geqq 1$); let $\psi_{s'}(s)=0$ if $s\leqq s'$, $\psi_{s'}(s)=(s-s')^{n-1}$ if $s>s'$. The author establishes that for $t\in T$,

$$k(s',t)=R[\psi_{s'}]=\begin{cases}\int_{-\infty}^{s'}(s-s')^{n-1}dg(s), & s'<t,\\ -\int_{s'}^{\infty}(s-s')^{n-1}dg(s), & s'\geqq t.\end{cases}$$

Suppose that $x(s)$ is a function whose derivative of order $n-1$ exists and is absolutely continuous on every finite s-interval. A necessary and sufficient condition that $R[x]$ and

$$R^*[x]=\int_{-\infty}^{\infty}x^{(n)}(s')k(s',t)ds'$$

exist and be equal is that the integral

$$I=\int_{-\infty}^{\infty}dg(s)\int_0^s(s-s')^{n-1}x^{(n)}(s')ds'$$

exist and that the order of integration in I be invertible, for all $t\in T$. Furthermore, the author gives sufficient conditions for $R[x]=R^*[x]$. *S. C. van Veen* (Delft).

Mandelbrojt, S. Théorèmes d'unicité. Ann. Sci. École Norm. Sup. (3) 65, 101–138 (1948).

The author's "fundamental inequality" [same Ann. (3) 63, 351–378 (1936); these Rev. 9, 229] is applied to various uniqueness problems. Notation: $\{\lambda_n\}$ is an increasing sequence of natural numbers, $\{k_n\}=\{1, 2, 3, \cdots\}-\{\lambda_n\}$ is the complementary sequence; $N(x)$ is the number of $\lambda_n<x$,

$$d(\lambda)=\sup_{x\geqq\lambda} x^{-1}\int_0^x t^{-1}N(t)dt, \quad d=\lim_{\lambda\to\infty} d(\lambda);$$

$p(\sigma)$ is an increasing, positive function. The condition $U(\lambda_n, p(\sigma), a)$ states that there are a constant $a>d$ and a $\gamma>0$ such that

$$\int^\infty p(\sigma) \exp\left\{-\frac{1}{2}\int^\sigma [a-d(\gamma p(u)]^{-1}du\right\}d\sigma=\infty.$$

Typical results are the following. (1) Let $F(x)$ be an even continuous positive function such that $p(\sigma)=\log F(e^\sigma)$ is a convex function of σ. If $U(\lambda_n, p(\sigma), \frac{1}{2})$ is true, then $\{x^{k_n}/F(x)\}$ is closed on $(-\infty, \infty)$ in the L^p-sense $(1\leqq p\leqq\infty)$. (2) Same hypothesis as in (1) and $F^{(k_1)}(x)=O((F(x)^{1+1/k_1})$. Let $f(x)$ be a continuous function satisfying $f(x)/F(x)\to 0$ as $|x|\to\infty$. Given $\epsilon>0$ there is a $P(x)=\sum_{k=1}^N a_n x^{k_n}$ such that $|f(x)-P(x)|<\epsilon F(x)$. This contains a theorem of S. Bernstein as a very special case. (3) Let $K(x, y)$ $(a\leqq x, y\leqq b)$ be a real, symmetric kernel such that the iterated kernels $K^{(n)}(x, y)$ $(n=1, 2, \cdots)$ exist and satisfy

$$\int_a^b (K^{(n)}(x, y))^2dy<\infty$$

except perhaps for an enumerable set $E=\{\xi\}$ of x. Suppose that x_0, y_0 do not belong to the closure of E. Put $M_{n-1}^2=K^{(2n)}(x_0, x_0)K^{(2n)}(y_0, y_0)$, $p(\sigma)=\sup_{n\geqq 1}(2n\sigma-\log M_n)$. If $k_1=2$, then $U(\lambda_n, p(\sigma), \frac{1}{2})$ and $K^{(k_n)}(x_0, y_0)=0$ $(n=1, 2, \cdots)$ imply $K^{(p)}(x_0, y_0)=0$ for $p\geqq 2$. (4) If

$$p(\sigma)=\sup_{n\geqq 1}(k_n\sigma-\log M_n)$$

and $U(\lambda_n, p(\sigma), 1)$ is true, then the moment problem $\int_0^\infty t^{k_n}dV=M_n$ $(V\uparrow)$ is determined. If

$$p(\sigma)=\sup_{n\geqq 1}(k_{2n}\sigma-\log M_{2n})$$

and $U(\lambda_n, p(\sigma), \frac{1}{2})$ is true, then $\int_{-\infty}^\infty t^{k_n}dV=M_n$ $(V\uparrow)$ is a determined moment problem. Carleman's conditions for the uniqueness of the moment problem are contained in this theorem as special cases. *W. H. J. Fuchs.*

Brunk, H. D. A consistency theorem. Bull. Amer. Math. Soc. 55, 204–212 (1949).

Proofs of results previously announced [C. R. Acad. Sci. Paris 226, 460–462 (1948); these Rev. 9, 416].

W. H. J. Fuchs (Liverpool).

Calculus

***Timofeev, A. F. Integrirovanie Funkciĭ. [Integration of Functions].** OGIZ, Moscow-Leningrad, 1948. 432 pp.

This is a compilation of elementary methods for evaluating indefinite integrals of elementary functions. Numerous illustrative examples are given. *R. P. Boas, Jr.*

Ballantine, J. P. Relative infinitesimals. Univ. Washington Publ. Math. 2, no. 3, 5–27 (1940).

Analytic functions of x, y are considered, expressed in series in powers of x and y. The function $F(x, y)$ is an absolute infinitesimal of order equal to the degree of the lowest degree terms in the series. Given $I(x, y)$ and $J(x, y)$, absolute infinitesimals of first order, if infinitesimals $I'(x, y)$, $J'(x, y)$ can be found of absolute order zero, so that $II'+JJ'$ is an absolute infinitesimal of order n, then I is a relative infinitesimal of order (at least) n relative to J. If I is of absolute order greater than 1, its order relative to J is defined to be that of $I+J$. [There seems to be some difficulty in proving theorem 2 below, which is easily proved if the condition "J' of absolute order zero" is removed. In the important case that I is of absolute order 1, the condition is a consequence of the condition for I'. The author, in a communication to the reviewer, has proposed dropping the condition for J'.] Theorem 1. If I and I^* are infinitesimals of orders i and i^* relative to J, then II^* is of order $i+i^*$ relative to J. Theorem 2. If $i<i^*$, then $I+I^*$ is of order i relative to J; if $i=i^*$, $I+I^*$ is of order at least i relative to J. Theorem 3. If I and J are two absolute infinitesimals of first order, and if I is of order i relative to J, then the curves $I=0$ and $J=0$ both pass through the origin, and have i-point contact at that point. Most theorems are stated without proof.

The author shows how to use relative infinitesimals in finding tangent lines; finding osculating circles; solving $J(x, y)=0$ for y; finding osculating conics; finding osculating parabolas; study of the locus $F(x, y)=0$ near the origin when F is an absolute second order infinitesimal; finding osculating circles at a double point; approximating branches at a double point; cusps; asymptotes; approximation to $F(x, y)$ for large values of x and y; closeness of fit; curves passing close to, but not through, the origin; plotting complicated equations. Some of the methods involve algorithms. The object, in each case, is to introduce a practicable method of obtaining a result which would otherwise be obtained by laborious calculation. *A. B. Brown.*

Ramaswami, V. On polynomials and Lagrange's form of the general mean-value theorem. Bull. Amer. Math. Soc. 54, 946–949 (1948).

L'auteur démontre que si, dans le reste de la formule de Taylor pris sous la forme de Lagrange:

$$f(x+h)=f(x)+\cdots+(h^n/n!)f^{(n)}(x+\theta h),$$

$\theta(x, h)$ est un polynome en x, alors $f(x)$ est un polynome de degré $n+1$ au plus; il prouve que ce résultat vaut aussi avec des hypothèses plus générales. *J. Favard* (Paris).

Heffter, Lothar. Differentiation und Integration bestimmter Integrale nach einem Parameter. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 3, 113–119 (1948).

Considérations élémentaires sur les conditions suffisantes pour la validité des formules

$$\frac{d}{dy}\int_a^b f(x, y)dx=\int_a^b \frac{\partial f}{\partial y}dx,$$

$$\int_\alpha^\beta dy\int_a^b f(x, y)dx=\int_a^b dx\int_\alpha^\beta f(x, y)dy,$$

illustrées par des exemples. *A. Ghizzetti* (Pisa).

Theory of Sets, Theory of Functions of Real Variables

Kurepa, Georges. Sur les ensembles ordonnés dénombrables. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 3, 145–151 (1948). (French. Croatian summary)

The two following theorems are established. In order that a series E should be ordinally similar to a subseries of η (the rationals in natural order) and that η should be ordinally similar to a subseries of E it is necessary and sufficient that E should be countable and that $\Gamma(E)=\Omega$, where $\Gamma(E)$ is the upper bound of the ordinals of well-ordered subseries of E and where Ω is the least ordinal of the third class. In order that E should be at most countable and ordinally scattered (i.e., contain no ordinally dense subseries) it is necessary and sufficient that $\Gamma(E)+\Gamma(E^*)<\Omega$ where E^* is the series E in inverse order. *J. Todd* (London).

Sikorski, Roman. On the representation of Boolean algebras as fields of sets. Fund. Math. 35, 247–258 (1948).

The author studies conditions on an $\mathfrak{m}$-complete Boolean algebra $\mathbf{A}$ in order that it be representable as an $\mathfrak{m}$-additive field of sets, where $\mathfrak{m}$ is an infinite cardinal number. This review will be limited to three indicative results. (1) In case $\mathbf{A}$ is a quotient algebra $\mathbf{X}/\mathbf{I}$ of an $\mathfrak{m}$-additive algebra of sets $\mathbf{X}$ by an $\mathfrak{m}$-additive ideal $\mathbf{I}$, it is shown that a sufficient condition is that $\mathbf{I}$ be semi-principal, that is, that $\mathbf{I}$ be the ideal of all sets of $\mathbf{X}$ which are subsets of a fixed set X_0 (not necessarily in $\mathbf{X}$) and that this condition is also necessary if every two-valued $\mathfrak{m}$-additive measure on $\mathbf{X}$ is trivial (i.e., is concentrated at a point). (2) A σ-complete algebra $\mathbf{A}$ is said to satisfy the condition (M) if whenever a nonzero element A is broken down dyadically, e.g., as in the breakdown of the unit interval into halves, quarters, eighths, etc., there is at least one decreasing sequence with nonvoid intersection. The condition (M) is necessary in order that $\mathbf{A}$ be isomorphic to a σ-field of sets, but the author shows by a counter-example that it is not sufficient. (3) The author gives another proof, similar to that announced by Halmos [Bull. Amer. Math. Soc. 54, 1083 (1948)] of the theorem that if $\mathfrak{m}=\aleph_0$ then every $\mathfrak{m}$-complete Boolean algebra $\mathbf{A}$ is realizable as a quotient $\mathbf{X}/\mathbf{I}$, where $\mathbf{X}$ is an $\mathfrak{m}$-additive field of sets and $\mathbf{I}$ is an $\mathfrak{m}$-additive ideal in $\mathbf{X}$, and he shows by a counter-example that the theorem is false if $\mathfrak{m}\geqq 2^{\aleph_0}$.
L. H. Loomis (Cambridge, Mass.).

Kappos, Demetrios A. Ein Beitrag zur Carathéodoryschen Definition der Ortsfunktionen in Booleschen Algebren. Math. Z. 51, 616–634 (1949).

The author constructs the analogues of point functions in a Boolean algebra whose elements are not necessarily sets by starting with the natural analogues of simple functions and then completing the system by Dedekind cuts.
P. R. Halmos (Chicago, Ill.).

Andersen, Erik Sparre, and Jessen, Børge. Some limit theorems on set-functions. Danske Vid. Selsk. Mat.-Fys. Medd. 25, no. 5, 8 pp. (1948).

If E is a measure space with a measure μ such that $\mu(E)=1$, if f is an integrable function on E, and if, for every measurable set A,

$$\varphi(A)=\int_A f d\mu+\mu(A\cap\{x:f(x)=+\infty\})-\mu(A\cap\{x:f(x)=-\infty\}),$$

then the function f is called a derivative of the set function φ with respect to μ. The authors prove that if (1) $\{\mathfrak{F}_n\}$ is a monotone increasing [decreasing] sequence of σ-subalgebras of the σ-algebra of all measurable sets, (2) $\mathfrak{F}_0$ is the least σ-algebra containing all $\mathfrak{F}_n$ [the greatest σ-algebra contained in all $\mathfrak{F}_n$], (3) φ is a bounded countably additive set function on the class of all measurable sets, (4) φ_n and μ_n are the contractions of φ and μ to $\mathfrak{F}_n$, $n=0, 1, 2, \cdots$, and (5) f_n is a derivative of φ_n with respect to μ_n, $n=1, 2, \cdots$, then both the functions $f_*=\lim\inf f_n$ and $f^*=\lim\sup f_n$ are derivatives of φ_0 with respect to μ_0. *P. R. Halmos.*

Sierpiński, Wacław. Sur l'équivalence des ensembles par décomposition en deux parties. Fund. Math. 35, 151–158 (1948).

Two sets A and B in a Euclidean space are equivalent by decomposition into n parts (notation, $A={}_nB$) if they admit decompositions into disjoint components: $A=A_1+\cdots+A_n$, $B=B_1+\cdots+B_n$, with A_i and B_i, $i=1, \cdots, n$, congruent, i.e., superposable by translation or rotation. The sets A and B are equivalent by finite decomposition if there exists a positive integer n such that $A={}_nB$. Among other results, the author proves that, if K is the linear continuum, and B any bounded or denumerable set, then $K={}_2K-B$. If R, A, respectively, are the sets of rational and of algebraic numbers, then R and A are not equivalent by finite decomposition; nor are R and D so equivalent, where D is the set of finite decimal fractions. It is finally proved, with the aid of a result of Hausdorff, that the surface S of a sphere is decomposable into 10 disjoint parts of which four and six yield respectively, upon suitable translations and rotations, two spherical surfaces each congruent to S. This result connects with one due to R. M. Robinson [Fund. Math. 34, 246–260 (1947), in particular, p. 254; these Rev. 10, 106]. Most of the proofs are elementary and brief. *H. Blumberg* (Columbus, Ohio).

Sierpiński, Wacław. Sur un paradoxe de M. J. von Neumann. Fund. Math. 35, 203–207 (1948).

If A and B are two sets lying in a metric space M, then A is smaller than B in the sense of J. von Neumann ($A<_NB$) if there exists a function $b=f(a)$ on A to B ($a\epsilon A$, $b\epsilon B$), which mates A and B biuniquely, such that $\rho(f(p), f(q))<\rho(p, q)$ for $p\epsilon A$, $q\epsilon A$, $p\neq q$, ρ signifying the distance function for M. Say that $A<_NB$ by finite decomposition if there exist decompositions of A and B into disjoint components: $A=A_1+\cdots+A_n$, $B=B_1+\cdots+B_n$, such that $A_i<_NB_i$, $i=1, \cdots, n$. With the aid of a result of Banach and Tarski [Fund. Math. 6, 244–277 (1924), in particular, p. 267, théorème 31], the author proves that if C_1, C_2 are two planar circles (interior and boundary), whether congruent or not, then $C_1<_NC_2$ by finite decomposition.
H. Blumberg (Columbus, Ohio).

Besicovitch, A. S., and Miller, D. S. On the set of distances between the points of a Carathéodory linearly measurable plane point set. Proc. London Math. Soc. (2) 50, 305–316 (1948).

A set A in a metric space is said to have the property of Steinhaus if the distances $d(p, q)$ (formed for all points p, q in A) fill out an interval $(0, \delta)$. H. Steinhaus [Fund. Math. 1, 93–104 (1920)] showed that this property holds for every subset of the straight line with positive Lebesgue inner measure. If Carathéodory linear measure is used for plane sets, then, as the present authors show, positive inner

measure is not sufficient to ensure that the property of Steinhaus holds. However the property does hold if the plane set is also a subset of a rectifiable curve. From Besicovitch's investigations of densities [Math. Ann. **98**, 422–464 (1928); **115**, 296–329 (1938)] it follows that the Steinhaus property holds for every plane set which is linearly measurable, has finite positive measure, and is regular, that is, has upper and lower densities both equal to 1 at almost all points of the set.

By modifying constructions previously given by Gross, Besicovitch and Sierpiński, the authors exhibit examples of linearly measurable plane sets of finite positive measure (but not regular) as follows: A_1 has distances which fill up an interval but do not fill up any interval with the origin as end-point; A_2 has lower density $c>0$ at almost every one of its points, yet does not have the property of Steinhaus; A_3 has lower density not exceeding $\frac{1}{2}$ at every one of its points, yet does have the property of Steinhaus.

I. Halperin (Kingston, Ont.).

Besicovitch, A. S. On distance-sets. J. London Math. Soc. **23**, 9–14 (1 plate) (1948).

Let C be a bounded, continuous plane curve, decomposed into mutually exclusive linearly measurable subsets A_1, A_2. The results of Besicovitch and Miller [see the preceding review] imply: (1) if C is rectifiable, then A_1 will certainly have the property of Steinhaus if its measure $m(A_1)>0$, in other words, if $m(A_2)<m(C)$; (2) if C is not rectifiable, A_1 may fail to have the property of Steinhaus even though $m(A_1)>0$. Of course, in (2), A_2 cannot be empty. In the results of Besicovitch and Miller, (2) is shown only with $m(A_1)$ finite, and hence with $m(A_2)$ infinite. In the present paper examples are constructed of (2) with $m(A_2)$ finite. (A suitable contraction of the plane then gives (2) with $m(A_2)$ less than any previously assigned positive number.)

I. Halperin (Kingston, Ont.).

Mossaheb, G. H. On the problem of the set of distances. J. London Math. Soc. **22** (1947), 252–256 (1948).

From the results of Besicovitch and Miller [see the preceding two reviews] it follows that a plane set of finite positive Carathéodory α-measure has the property of Steinhaus if (i) $\alpha=2$ or (ii) $\alpha=1$ and the set is also regular. In the present paper it is shown that for $1<\alpha<2$, an additional condition (which is a generalization of regularity) is not sufficient. For every α with $1<\alpha<2$, the author constructs a perfect set P which fails to have the property of Steinhaus but does have positive finite α-measure, has positive lower α-density at each of its points, and can be included in a simple curve C such that C also has positive lower α-density at each of its points and the α-measure of C is arbitrarily close to that of P.

I. Halperin (Kingston, Ont.).

Goffman, Casper. Proof of a theorem of Saks and Sierpiński. Bull. Amer. Math. Soc. **54**, 950–952 (1948).

Proof, without use of the corresponding result for measurable functions, of the following theorem of Saks and Sierpiński. If $f(x)$ is any real function defined in the closed interval $I:(0, 1)$, there exists a function $g(x)$ of Baire's class less than or equal to 2 such that, for every $\epsilon>0$, the set on which $|f(x)-g(x)|<\epsilon$ is of exterior measure 1. The argument is based on the following lemma. If $f(x)$ is a real function defined in I, $\epsilon>0$, $g(x)$ is a continuous function on I, and the set where $|f(x)-g(x)|<\epsilon$ is of exterior measure greater than $1-\epsilon$, then for every $\eta>0$, there exists a continuous function $c(x)$ such that the set where $|g(x)-c(x)|<\epsilon$ is of measure greater than $1-\epsilon$, and the set where $|f(x)-c(x)|<\eta$ is of exterior measure greater than $1-\eta$.

H. Blumberg.

Froda, Alexandre. Sur la réoscillation de voisinage des fonctions de variables réelles. C. R. Acad. Sci. Paris **227**, 1200–1201 (1948).

Let $f(x)$ be a function of n variables, defined in an interval I, and designate by $\omega(f, a)=\limsup_{x\to a} f(x)-\liminf_{x\to a} f(x)$ the oscillation of f at a (hence not taking the value $f(a)$ into consideration). The author defines, by transfinite induction, the oscillation $\omega^\alpha(f, a)=\omega(\omega^{\alpha-1}, a)$ for α an ordinal number of the first kind (setting $\omega^0(f, a)=\omega(f, a)$), and $\omega^\alpha(f, a)=\lim \omega^{\alpha'}(f, a)$ with $\alpha'<\alpha$ and $\alpha'\to\alpha$ for any α of the second kind, where the unique existence of this limit can be proved. The author states the following theorem. For a given $f(x)$ there exists an $\bar{\alpha}$ such that $\omega^\alpha(f, x)=\omega^{\alpha+1}(f, x)$ for $\alpha\geqq\alpha'$ and for all $x\epsilon I$, while for every $\alpha<\bar{\alpha}$ one has $\omega^\alpha(f, x_\alpha)>\omega^{\alpha+1}(f, x_\alpha)$ at least at one point $x_\alpha\epsilon I$. The function $\omega^{\bar{\alpha}}(f, x)$ is called "réoscillation (de voisinage)" of $f(x)$ by the author. Moreover, he states that for a given $\bar{\alpha}$ functions $f(x)$ with a "réoscillation" corresponding to $\bar{\alpha}$ can effectively be constructed. Only indications of the proofs are given. [Reviewer's remark. The author's results are closely related to results (concerning "saltus-functions," in case a finite number of points may be neglected) obtained by H. Blumberg, Proc. Nat. Acad. Sci. U. S. A. **2**, 646–649 (1916); Ann. of Math. (2) **18**, 147–160 (1917)].

A. Rosenthal (Lafayette, Ind.).

Behrend, F. A. Some remarks on the construction of continuous non-differentiable functions. Proc. London Math. Soc. (2) **50**, 463–481 (1949).

In generalisation of the nondifferentiable functions of Weierstrass and others the author considers

$$f(x)=\sum_{r=0}^{\infty}a^r g(b^r x),$$

where $0<a<1$, and $g(x)$ is continuous and bounded for all x, and is periodic (or, more generally, "pseudo-periodic" in a special sense). He proves by elementary arguments (i) that, if $g(x)$ also satisfies a Lipschitz condition and ab exceeds a certain number depending on $g(x)$, then $f(x)$ has not a derivative, finite or infinite, at any point (in particular, for Weierstrass's function, with $g(x)=\cos \pi x$, the condition $ab\geqq 7.05$ is sufficient); (ii) that, if $g(x)$ satisfies a "second order" Lipschitz condition, $ab\geqq 1$ and ab exceeds a certain number, then $g(x)$ has not a finite derivative at any point (for Weierstrass's function the condition is $b\geqq 20/3$, which is weaker than known results). He points out also that, with certain restrictions, if $g(x)$ does not satisfy the condition of (ii) but is "piecewise linear" then, for b an integer and $ab\geqq 1$, $f(x)$ may itself be piecewise linear for certain values of a and b, but for all other values has not a finite derivative at any point.

U. S. Haslam-Jones (Oxford).

Scorza Dragoni, G. Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile. Rend. Sem. Mat. Univ. Padova **17**, 102–106 (1948).

Let the finite-valued, real-valued function $f(x, y)$ be defined in the rectangle R: $a\leqq x\leqq b$, $c\leqq y\leqq d$, and assume that $f(x, y)$ is measurable with respect to x and continuous with respect to y. Let $\epsilon>0$ be assigned. Then there exists a perfect subset i of the interval $a\leqq x\leqq b$ such that $f(x, y)$ is uniformly continuous on the set for which $(x, y)\epsilon R$, $x\epsilon i$, and

such that the (linear) measure of i exceeds $b-a-\epsilon$. The proof is based on a lemma of Cesari, according to which the maximum (and also the minimum) of $f(x, y)$ on $c \leqq y \leqq d$ is a measurable function of x. *T. Radó* (Columbus, Ohio).

Baiada, Emilio. Sulle funzioni continue separatamente rispetto alle variabili e gli integrali curvilinei. Rend. Sem. Mat. Univ. Padova **17**, 201–218 (1948).

The purpose of the paper is to study the following problem stated by Scorza Dragoni [see the preceding review]. In the square Q: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, let $f(x, y)$ be a real-valued function which is continuous with respect to x and y separately. Let n be a positive integer, and let Δ_n^x denote a plurinterval (sum of a finite or countably infinite set of intervals) on the x-axis, with length-sum less than $1/n$. Let Δ_n^y have similar meaning relative to y. Question: is it possible, for every n, to choose Δ_n^x and Δ_n^y in such a manner that $f(x, y)$ is continuous with respect to (x, y) on Q minus the Cartesian product $\Delta_n^x \times \Delta_n^y$? The author proves that this is indeed the case. Application is made to the classical Looman-Menchoff theorem concerning the Cauchy-Riemann equations. *T. Radó* (Columbus, Ohio).

Fiori, Anna. Sull'integrazione delle funzioni d'insieme. Matematiche, Catania **3**, 68–91 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **213** (1948).

The author extends the definition and the elementary properties of Burkill integrals, defined for functions of intervals, to functions defined on certain classes of Lebesgue measurable sets. *P. R. Halmos* (Chicago, Ill.).

Radó, Tibor. Convergence in area. Duke Math. J. **16**, 61–71 (1949).

Let S_n, S, Σ_n, Σ denote parametric Fréchet surfaces defined by triplets of real functions $x_n, y_n, z_n; x, y, z; x_n, y_n, 0; x, y, 0$ of a point (u, v) of the unit square. Suppose further that $S_n \rightarrow S$ and that $A(S_n) \rightarrow A(S) < \infty$, where A denotes the Lebesgue area. Then $A(\Sigma_n) \rightarrow A(\Sigma)$. In a note added in proof the author remarks that the theorem has also been proved by L. Cesari [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 486–489, 489–495 (1947); these Rev. **10**, 240] by other methods. The previous best result in this direction is due to P. V. Reichelderfer [Trans. Amer. Math. Soc. **53**, 251–291 (1943); these Rev. **4**, 213]. Here the author deduces the theorem by a simple polyhedral approximation argument from one of the main results of his book, viz., from the equality of his "lower area" with the Lebesgue area in the case in which the latter is finite. *L. C. Young* (Madison, Wis.).

Helsel, R. G. Convergence in area and convergence in volume. Duke Math. J. **16**, 111–118 (1949).

Let $A(S)$ denote the Lebesgue area of the closed Fréchet oriented surface S situated in space of (x, y, z), and let $|S|$ denote the 3-dimensional measure of the set of points situated on S. Further let $i(x, y, z)$ be the topological index of (x, y, z) with respect to S, where $i(x, y, z)=0$ when (x, y, z) lies on S, and let $V(S)=\iiint |i(x, y, z)|\,dxdydz$ be termed the volume enclosed by S. With these definitions it is proved that the hypotheses $S_n \rightarrow S$, $|S|=0$, $A(S_n) \rightarrow A(S) < \infty$ imply $V(S_n) \rightarrow V(S)$. The author deduces this theorem from the theorem of the preceding review, by methods which have some similarity with those used by T. Radó in the corresponding problem for areas enclosed by plane curves [Fund. Math. **27**, 212–225 (1936)]. *L. C. Young*.

Theory of Functions of Complex Variables

***Behnke, H. Klassische Funktionentheorie.** Aschendorff'sche Verlagsbuchhandlung, Münster. Teil I, 1947, v+299+xli pp., 8 Marks; Teil II, 1948, iii+217+xvi pp., 6 Marks.

These volumes (volume II with the collaboration of F. Sommer) contain the author's lectures on the subject. The first volume culminates in a proof of the Riemann mapping theorem; the appendices contain, among other things, E. Schmidt's proof of the Jordan curve theorem and a proof of the transcendence of π. The second volume contains a detailed discussion of Riemann surfaces, the uniformization theorem and related matters (such as conformal mapping on the boundary and automorphic functions). The exposition is uniformly thorough and clear. *R. P. Boas, Jr.*

***Churchill, Ruel V. Introduction to Complex Variables and Applications.** McGraw-Hill Book Company, Inc., New York, 1948. vi+216 pp. $3.50.

An elementary text for a one semester course. Applications are restricted to the evaluation of integrals by residues and to certain uses of conformal mapping. *N. Levinson* (Cambridge, Mass.).

Jacob, Caius. Sur un problème mixte pour le plan muni de coupures rectilignes alignées. C. R. Acad. Sci. Paris **228**, 355–357 (1949).

L'auteur considère sur l'axe réel du plan complexe des coupures a_ib_i et c_kd_k et cherche la fonction holomorphe $F(z)=U+iV$ telle que U admette des valeurs-limites données sur les deux bords des a_ib_i et V des valeurs-limites données sur ceux des c_kd_k. Il donne les conditions d'uniformité de $F(z)$ et indique que la solution générale se ramène à celles de deux cas particuliers où on les explicite simplement. *M. Brelot* (Grenoble).

Magnaradze, L. G. On a linear boundary problem of the theory of functions of a complex variable. Doklady Akad. Nauk SSSR (N.S.) **64**, 17–20 (1949). (Russian)

Let D^+ be the bounded domain in the complex plane, bounded by a simple closed "smooth" curve L; D^-, the complement of $D^+ + L$. The author studies the problem of finding vectors $\phi^+(z)=(\phi_1^+(z), \cdots, \phi_n^+(z))$, analytic in D^+, and $\phi^-(z)=(\phi_1^-(z), \cdots, \phi_n^-(z))$, analytic in D^- (of finite order of infinity at ∞), so that $\phi^- = C\phi^+$ (on L); here $C(t)=(c_{ij}(t))$ $(i, j=1, \cdots, n)$ is a matrix assigned on L; $\det C(t) \neq 0$ (on L). On letting $\omega(\tau; c_{ij}) = \max |c_{ij}(t_2)-c_{ij}(t_1)|$ $(|s_2-s_1| \leqq \tau$; s being length along $L)$, it is assumed that $\omega(\tau; c_{ij})$ is $o(\log^{-p} \tau^{-1})$ (all $p \geqq 0$), except for a finite number of discontinuities of the first kind; in unilateral neighborhoods of each of the latter it is supposed that $\omega(\tau; c_{ij}) = O(\tau^{\mu})$ $(\sigma < \mu \leqq 1)$, where σ is a suitable nonnegative number. The solution of this problem constitutes an extension to less restrictive conditions of some known results in the field of Riemann-Hilbert boundary problems. *W. J. Trjitzinsky* (Urbana, Ill.).

Weigand, Leonhard. Über die Randwerte meromorpher Funktionen einer Veränderlichen. Comment. Math. Helv. **22**, 125–149 (1949).

Carathéodory proved [same Comment. **19**, 263–278 (1946); these Rev. **8**, 508] a theorem, whose inversion the author establishes: given a decomposition $H+\sum G_i$ of the complex plane into H, a closed set, and a series of non-overlapping open sets G_i, then there exists an $f(z)$, mero-

morphic in $|z|<1$, such that at a boundary point ζ, $|\zeta|=1$, it admits boundary values which constitute $H+\sum' G_i$. Here $\sum' G_i$ denotes the sum of a fixed arbitrary number of the given G_i's which includes all those that are multiply connected. The proof is too complicated to be discussed here.

František Wolf (Berkeley, Calif.).

Chazy, Jean. Sur le rayon de convergence de la série de Lagrange. C. R. Acad. Sci. Paris **228**, 613–616 (1949).

La série de Lagrange donne le développement en série entière en α de la solution $z(\alpha)$ de l'équation $z=a+\alpha f(z)$ qui est égale à a pour $\alpha=0$. On suppose $f(z)$ holomorphe pour $|z-a|<R$. Rouché a montré [J. École Polytech. **22**, cahier **39**, 193–224 (1862)] que, si $M(\rho)$ est le maximum de $|f(z)|$ pour $|z-a|\leqq\rho$, la série de Lagrange converge certainement pour $|\alpha|<r$, r étant le maximum de $\rho/M(\rho)$ pour $\rho<R$. Rouché établit ensuite que la circonférence $|\alpha|=r$ contient au moins un point critique algébrique de la branche $z(\alpha)$ considérée; r serait donc rigoureusement égal au rayon de convergence. L'auteur reprend la démonstration de ce second résultat de Rouché par une méthode un peu différente, et vérifie en détail le résultat pour l'équation de Kepler. Il observe que, a priori, le rayon de convergence aurait pu être déterminé par une singularité transcendante de la branche $z(\alpha)$. [On peut remarquer que l'auteur admet, p. 614, que $\theta=\varphi(\rho)$ étant la courbe le long de laquelle $|f(z)|=M(\rho)$ $(\arg(z-a)=\theta,\ |z-a|=\rho)$, la dérivée $\varphi'(\rho)$ existe quel que soit ρ. Or, Blumenthal, Bull. Soc. Math. France **35**, 213–232 (1907), et Hardy, Quart. J. Math. **41**, 1–9 (1909), ont montré qu'il n'en est pas toujours ainsi, même lorsque $f(z)$ est une fonction entière. En se bornant à $f(z)$ entière et au cas canonique $a=0$, on voit que la fonction inverse $z(\alpha)$ de $\alpha=z/f(z)$ peut ne pas posséder de singularités algébriques, l'équation $f(z)-zf'(z)=0$ pouvant ne pas avoir de racines; dans ces cas, le résultat de Rouché est en défaut.]

G. Valiron (Paris).

Consiglio, A. Determinazione in termini finiti di rappresentazioni conformi. Matematiche, Catania **3**, 50–58 (1948).

Nehari, Zeev. The kernel function and canonical conformal maps. Duke Math. J. **16**, 165–178 (1949).

Let D denote a finite multiply connected region in the complex z-plane. The author obtains expressions for the mapping functions of D onto various types of canonical domains in terms of the Bergman kernel function $K(z,\bar\zeta)$ of D [Math. Ann. **86**, 238–271 (1922)]. Thus, if the boundary Γ of D consists of closed analytic curves and $w=\varphi_\theta(z)$ is an analytic function which possesses a simple pole with residue 1 at $z=\zeta(\zeta\varepsilon D)$ and maps D on the full w-plane with parallel rectilinear slits forming a given angle θ with the positive axis of reals, then

$$\varphi_\theta'(z)=-\frac{1}{(z-\zeta)^2}-ie^{i\theta}\int_\Gamma\frac{\Re\{e^{i\theta}K(t,\bar\zeta)\}dt}{t-z}.$$

The author indicates how the condition on Γ can be relaxed. Somewhat similar results are obtained for functions which map D (1) on the full plane with circular slits centered at the origin; (2) on the interior of a circle about the origin with concentric circular slits; (3) on the full plane with n circular holes, assuming D to be n-fold connected. In case (3), the Schwarzian derivative $\{w,z\}$ of the mapping function $w(z)$ is represented in terms of the kernel function $K(z,\bar\zeta)$ and the functions $w_k(z)$ whose real part is the harmonic measure $\omega_k(z)$ of D with respect to the kth boundary curve. Then, $w(z)$ can be obtained as the quotient of two linearly independent solutions of the differential equation $y''+\frac{1}{2}\{w,z\}y=0$. [In (27), β_ν is equal to $-K''(z_\nu,\bar\zeta)/\{4\pi[K'(z_\nu,\bar\zeta)]^2\}$. In (30), the equation should read (1) $U'dz=\overline{(V'dz)}-2\pi il^{-1}ds$, where l is the length of Γ. This correction necessitates a modification of the proof, pointed out to the reviewer by the author. If z_ν are the zeros of M', define U_1 and V_1 by (2) $U_1=\sum\gamma_\nu U(z,z_\nu)$, $V_1=\sum\bar\gamma_\nu V(z,z_\nu)$, where the γ_ν are the residues of $\lambda(z)$ at the points z_ν (see (26)). By (1) and (2), we have (3) $U_1'dz=\overline{V_1'dz}-2\pi il^{-1}(\sum\gamma_\nu)ds$. Since the expression within the parentheses on the right side of (24) is regular, the integral of the left side taken over Γ vanishes, whence by (25) and (26), $\sum\gamma_\nu=0$. Hence, (3) can be replaced by $U_1'dz=\overline{V_1'dz}$ and the argument of the paper can be repeated for U_1 and V_1 rather than U and V.] Extensive use is made of the "reproducing" property of $K(z,\bar\zeta)$.

W. Seidel (Los Angeles, Calif.).

Nehari, Zeev. On the accessory parameters of a Fuchsian differential equation. Amer. J. Math. **71**, 24–39 (1949).

Die konforme Abbildung eines Kreisbogenpolygons, dessen Seiten einen gemeinsamen Orthogonalkreis besitzen, wird bekanntlich durch eine analytische Funktion geleistet, die als Quotient der Integrale einer linearen homogenen Differentialgleichung mit rationalen Koeffizienten darstellbar ist. Falls die Anzahl n der Seiten des Polygons grösser als 3 ist, enthält die Differentialgleichung $n-3$ reelle "accessorische" Parameter, die von den Bestimmungsstücken des Polygons in ziemlich komplizierter Weise abhängen. In der vorliegenden Abhandlung wird die genannte Abhängigkeit insbesondere im Falle $n=4$ eingehend untersucht.

P. J. Myrberg (Helsingfors).

Tsuji, Masatsugu. On Löwner's differential equation in the theory of univalent functions. Jap. J. Math. **19**, 321–341 (1947).

Let L_{0,τ_0} be a Jordan arc $x=x(\tau)$, $0\leqq\tau\leqq\tau_0$, $0<|x(\tau)|<1$ for $\tau<\tau_0$, $|x(\tau_0)|=1$. Let D_τ be the simply connected region bounded by $|x|=1$ and by L_{τ,τ_0}, $L_{\tau_1,\tau}$ $(\tau_1<\tau)$ being that part of L_{0,τ_0} between $x(\tau_1)$ and $x(\tau)$. Let $x=x_\tau(W)$ $(x_\tau(0)=0$, $x_\tau'(0)>0)$ map D_τ on $|W|<1$, carrying $L_{0,\tau}$ into the Jordan arc $l_{0,\tau}$, the latter meeting $|W|=1$ at $e^{i\theta_\tau}$. Then $x=x_\tau(W)$ maps D_0 on Δ_τ, the region bounded by $l_{0,\tau}$ and by $|W|=1$. Let $W=W_\tau(z)$ $(W_\tau(0)=0,\ W_\tau'(0)=e^{-t}>0,\ t=t(\tau),\ t_0=t(\tau_0))$, $W_\tau(z)=f(z,t)$, $\theta_\tau=\theta(t)$, $f(z,0)\equiv z$. Then $\theta(t)$ is continuous for $0\leqq t\leqq t_0$ and $f(z,t)$ is continuous in z and t for $|z|\leqq1$, $0\leqq t\leqq t_0$, satisfying Löwner's differential equation

$$\frac{\partial}{\partial t}f(z,t)=-f(z,t)\frac{1+e^{-i\theta(t)}f(z,t)}{1-e^{-i\theta(t)}f(z,t)},$$

for $|z|<1$, $0\leqq t\leqq t_0$. Y. Komatu [Proc. Imp. Acad. Tokyo **17**, 11–17 (1941); these Rev. **2**, 276] has shown that, if L_{0,τ_0} is an analytic arc, $x(\tau)$ regular, $x'(\tau)\neq0$ for $0\leqq\tau\leqq\tau_0$, which meets $|x|=1$ orthogonally at $x(\tau_0)$, then $\theta'(t)$ exists and is continuous for $0<t<t_0$, and $\theta'(t)=-3K(t)$ for $0<t<t_0$, where $K(t)$ is the curvature of $l_{0,\tau}$ $(\tau=\tau(t))$ at $e^{i\theta(t)}$ and $K(t)>0$ if the part of $l_{0,\tau}$ in the vicinity of $e^{i\theta(t)}$ lies in the half-plane $\arg w>\theta(t)$ $(K(t)<0$ in the other case). This theorem of Komatu is extended by the author to the closed interval $0\leqq t\leqq t_0$, $K(0)$ being defined as $\lim_{t\to+0}K(t)$.

M. S. Robertson (New Brunswick, N. J.).

Myrberg, P. J. Über die analytische Fortsetzung von beschränkten Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **58**, 7 pp. (1949).

The author comments on various classes of sets which are removable from the function-theoretic point of view. An example is given where continua are removable. In fact, if F is a transcendental hyperelliptic Riemann surface defined by $y^2=g(x)$, g entire with simple zeros, and C is a simple closed circular disc lying wholly in one sheet of F, then there exists no nontrivial bounded analytic function with domain $F-C$. The proof depends on the observation that the zeros of a nonconstant analytic function do not cluster at a point of regularity. *M. Heins.*

Grunsky, H. Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten. III. Ein Einzigkeitssatz. Math. Z. **51**, 586–615 (1949).

[For part II cf. Jber. Deutsch. Math. Verein. **52**, 118–132 (1942); these Rev. **4**, 270.] The author continues his investigations concerning extremal problems for bounded analytic functions. In the earlier papers he was unable to settle the essential uniqueness of the extremals for one of the extremal problems considered. In the present paper he establishes uniqueness in the case where the domain of the competing functions is triply-connected. *M. Heins.*

Rajagopal, C. T. Carathéodory's inequality and allied results. II. Math. Student **15** (1947), 5–7 (1948).

[For part I cf. Math. Student **9**, 73–77 (1941); these Rev. **3**, 201.] Let a_0, R, M be given and let $\mathfrak{M}$ be the class of functions $f(z)=a_0+a_1z+\cdots$, satisfying $|f(z)|<M$, $|z|<R$. The author deduces the sharp bounds for $|a_1|$ and $|f(re^{i\theta})|$ for $f(z)\epsilon\mathfrak{M}$ from Schwarz's lemma. The sharp bound for $|a_n|$, $n=h>1$, follows, since $F(z)=a_0+a_hz/R^{h-1}+a_{2h}(z/R^{h-1})^2+\cdots$ also belongs to $\mathfrak{M}$, and a bound for $|f^{(\nu)}(z)|$ follows by addition. *W. K. Hayman* (Exeter).

Beckenbach, E. F., Gustin, W., and Shniad, H. On the mean modulus of an analytic function. Bull. Amer. Math. Soc. **55**, 184–190 (1949).

Suppose that the function $f(z)$ is regular in $|z|<1$. It is well known that, for every fixed t with $0\leqq t\leqq\infty$, the mean value

$$M_t(r)=M_t(r;f)=\left\{(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^td\theta\right\}^{1/t}$$

is a continuous nondecreasing function of r, and that $\log M_t(r)$ is a convex function of $\log r$ for $0\leqq r<1$. The authors investigate the convexity of $M_t(r)$ as a function of r itself. Let $T=T(f)$ be, for fixed f, the set of all parameters t for which $M_t(r)$ is convex for $0\leqq r<1$. Then T is closed but need not contain all t. The following results are obtained. (I) The set T always contains the values $t=2/k$ $(k=1, 2, 3, \cdots)$ and their limiting value $t=0$. In particular, $M_0(r)$, $M_1(r)$, $M_2(r)$ are convex functions of r. (II) If f has at most k zeros in $|z|<1$, then T contains the interval $0\leqq t\leqq 2/k$. (III) If $f(0)=0$ then T contains all t. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Wilson, R. Densities of strongest growth near an essential singularity. J. London Math. Soc. **23**, 246–250 (1948).

If ψ has 1 for an essential singularity, of order ρ and finite type, and is otherwise regular, then $\psi(z)=\sum_0^\infty F(n)z^n$, where $F(z)=\sum a_nz^n/n!$ is entire, of order $\rho/(\rho+1)$ and finite type. Let $f(z)=\sum a_nz^{-n-1}$; then $f(1/z)$ is entire, of order ρ and finite type, and $|f(z)-\psi(e^{-z})|$ is bounded as $z\to 0$. In a previous paper [Macintyre and the author, same J. **22** (1947), 298–304 (1948); these Rev. **10**, 25] it was shown that the directions of strongest growth of $\psi(e^{-1/z})=\phi(z)$, and of $F(z)$, as $z\to\infty$, coincide. In the present paper, the author now shows that, if there are only a finite number of such directions, then on any one of them the actual locations of the points at which $|F(z)|$ and $|\phi(z)|$ are large correspond in a simple manner. The method consists in showing that this holds for the functions $F(z)$ and $f(1/z)$. *R. C. Buck* (Providence, R. I.).

Wright, E. M. The asymptotic expansion of integral functions and of the coefficients in their Taylor series. Trans. Amer. Math. Soc. **64**, 409–438 (1948).

Let $\psi(t)$ be the finite sum of terms of the form at^b, where $0\leqq\Re(b)<1$. If $c(t)$ is regular and satisfies

$$(1)\qquad c(t)\sim t^\beta e^{\psi(t)}(e/\kappa t)^{\kappa t}$$

(where $\Re(\kappa)>0$) in a certain sector S, then the integral function $f(x)=\sum_{n=0}^\infty c(n)x^n$ is asymptotic to $F(X)$ in a certain sector, where X denotes a particular value of x^ρ $(\rho=1/\kappa)$ and $F(X)=2^{\frac12}\pi^{\frac12}\kappa^{-1-\beta}X^{\frac12+\beta}e^{P(X)}$, where $P(X)-X-\psi(\rho^X)$ is a finite sum of terms of the form PX^q. If $\Re(\rho)<\frac12$, the author gives an asymptotic formula for $f(x)$ for all large x, provided S is suitably chosen, but if $\Re(\rho)\geqq\frac12$, there may be parts of the x-plane in which the asymptotic behaviour of $f(x)$ does not depend solely on that of $c(t)$ for large t. If (1) is true for all large positive integral values of t, the asymptotic formula for $f(x)$ holds in the neighborhood of a certain curve which behaves in distant parts of the plane rather like the equiangular spiral $\arg X=\arg\kappa$.

If the right hand side of (1) is multiplied by

$$B_1t^{\beta_1}+\cdots+B_Mt^{\beta_M},$$

the author gives under general conditions an asymptotic expansion for $f(x)$ with a rule for calculating the coefficients occurring in that expansion. Finally he deduces the asymptotic expansion for large n of $c_\lambda(n)$ from that of $c(n)$, where λ is any number and $f(x+\lambda)=\sum_{n=0}^\infty c_\lambda(n)x^n$. *J. G. van der Corput* (Amsterdam).

Shah, S. M. A note on lower proximate orders. J. Indian Math. Soc. (N.S.) **12**, 31–32 (1948).

De même que l'on peut définir et utiliser un ordre précisé $\rho(r)$ pour une fonction entière $f(z)$ d'ordre fini positif [Valiron, Ann. Fac. Sci. Univ. Toulouse (3) **5**, 117–257 (1914); Lectures on the General Theory of Integral Functions, Cambridge, 1923; Shah, Bull. Amer. Math. Soc. **52**, 326–328 (1946); ces Rev. **7**, 380; voir aussi pour une construction élémentaire, Valiron, Directions de Borel des Fonctions Méromorphes, Mémor. Sci. Math., no. 89, Gauthier-Villars, Paris, 1938, p. 25], l'auteur définit un ordre précisé inférieur et montre son utilisation dans la recherche du nombre des zéros et du rang du terme maximum du développement taylorien de $f(z)$. *G. Valiron.*

Korevaar, J. A simple proof of a theorem of Pólya. Simon Stevin **26**, 81–89 (1949).

The theorem is that an entire function of zero exponential type is a constant if it is bounded at the integers. The proof was also given in the lecture reviewed below. *R. P. Boas, Jr.* (Providence, R. I.).

Korevaar, J. Entire functions of exponential type. Math. Centrum Amsterdam. Rapport **ZW 1948-011**, 10 pp. (1948). (Dutch)

This is a lecture surveying particularly a theorem of Pólya [cf. the following review], Carlson's theorem, and generalizations. *R. P. Boas, Jr.* (Providence, R. I.).

Korevaar, J. Interpolatory methods applied to functions of exponential type. Math. Centrum Amsterdam. Rapport **ZW 1948-018**, 16 pp. (1948). (Dutch. English summary)

The author gives new proofs of several known results and uses his methods to obtain some new theorems. In particular, he can prove quite simply that a function $f(z)$ of exponential type in $x\geqq 0$, of type less than π on $x=0$, and such that $f(n)=O(e^{an})$ for $n=1, 2, \cdots$, also satisfies $f(x)=O(e^{ax})$ for $x\rightarrow+\infty$. If $f(z)$ is of type less than $k\pi$ on $x=0$ and $f(n), f'(n), \cdots, f^{(k-1)}(n)$ are bounded for integral n, then $f(x)$ is bounded for real x. Several results are given on functions bounded on several sequences of points, generalizing theorems of Levinson and Ganapathy Iyer. *R. P. Boas, Jr.* (Providence, R. I.).

Schwartz, Marie-Hélène. Sur les indices de ramification de M. Nevanlinna. C. R. Acad. Sci. Paris **228**, 45–46 (1949).

Complétant ses résultats antérieurs [mêmes C. R. **210**, 525–526 (1940); ces Rev. **1**, 307] l'auteur énonce, comme compléments aux résultats de R. Nevanlinna [Eindeutige analytische Funktionen, Springer, Berlin, 1936, pp. 296–305] et de Teichmüller [Deutsche Math. **3**, 621–678 (1938)] les faits suivants. Si V désigne l'indice de ramification moyenne de Nevanlinna [loc. cit.] d'une surface de recouvrement dont les points critiques se projettent tous en q points fixes de la sphère (indice obtenu par passage à la limite au moyen de surfaces d'approche), on peut avoir lim sup $V>2$, même dans le cas parabolique, et $V=2$, même dans le cas hyperbolique. *S. Stoïlow* (Bucarest).

Parreau, Michel. Sur le théorème de Collingwood-Cartan. C. R. Acad. Sci. Paris **227**, 1323–1325 (1948).

Ce travail se rattache aux travaux récents de Collingwood [mêmes C. R. **227**, 615–617, 709–711, 749–751, 813–815 (1948); ces Rev. **10**, 244, 363]. Dinghas [Math. Z. **45**, 20–24 (1939)] avait étendu les anciens résultats de Collingwood et H. Cartan au cas du défaut d'Ahlfors, en montrant que: si le domaine simplement connexe D de la sphère de Riemann est complètement intérieur à un domaine simplement connexe D_0 au-dessus duquel la surface de Riemann de la fonction méromorphe $f(z)$ ne présente que des disques dont le nombre de feuillets est inférieur à un nombre fixe, le défaut d'Ahlfors pour D est nul. Suivant la méthode de Collingwood, mais utilisant les théorèmes d'Ahlfors sur le recouvrement, l'auteur montre que: si le domaine simplement connexe D de la sphère de Riemann Σ est complètement intérieur à des domaines $D(r)$ dont la frontière est à une distance $\sigma(r)$ de D, et si les disques et langues de la surface $\Sigma(r)$ décrite sur Σ par $f(z)$ lorsque $|z|\leqq r$, ont un nombre de feuillets au plus égal à $P(r)$, le défaut d'Ahlfors pour D est au plus égal à

$$\limsup_{r\to\infty} \frac{P(r)L(r)}{2\sigma(r)S(r)},$$

$\pi S(r)$ étant l'aire de $\Sigma(r)$ et $L(r)$ la longueur de la frontière de $\Sigma(r)$. L'auteur en déduit aussi une borne pour le défaut de Nevanlinna du domaine D. *G. Valiron* (Paris).

Parreau, Michel. Variation du défaut d'Ahlfors avec l'origine du plan des z. C. R. Acad. Sci. Paris **227**, 1198–1199 (1948).

Example which shows that the value of the defect mentioned in the title may depend on the choice of the origin. *L. Ahlfors* (Cambridge, Mass.).

Wittich, Hans. Über eine Klasse meromorpher Funktionen. Arch. Math. **1**, 160–166 (1948).

In this paper relations are deduced which permit one to compute the order, defects and ramification indices of a meromorphic function which belongs to a Riemann surface whose branch-points project into a finite number of points and whose graph contains a finite number of periodic ends. The results are obtained by means of an auxiliary quasi-conformal mapping; essential use is made of a theorem due to Teichmüller for which the author has recently given a new proof [Math. Z. **51**, 278–288 (1948); these Rev. **10**, 241]. It is interesting that in this way functions are found whose order is a transcendental number while the defects are irrational algebraic numbers. *L. Ahlfors.*

Stoïlow, Simon. Quelques remarques sur les éléments frontière des surfaces de Riemann et sur les fonctions correspondant à ces surfaces. C. R. Acad. Sci. Paris **227**, 1326–1328 (1948).

The author introduces the notion of "élément frontière totalement étalé" of a covering surface of the z-plane. An élément frontière is a nested sequence of regions Δ_n on the surface which converges to "∞," and it is totalement étalé if the projection of each Δ_n is dense in the plane. The following results are announced. (1) A nonalgebroid surface with a zero boundary (in the sense of Nevanlinna) has at least one élément frontière totalement étalé; (2) the Riemann surface of an automorphism $w=w(z)$, defined by $f(w)=f(z)$, is either closed or possesses an élément frontière totalement étalé. *L. Ahlfors* (Cambridge, Mass.).

Radojčić, M. Remarque sur le problème des types des surfaces de Riemann. Acad. Serbe Sci. Publ. Inst. Math. **1**, 97–100 (1947).

It is remarked that the transcendental singularities of a simply connected Riemann surface of parabolic type can form an everywhere dense ordered set. *L. Ahlfors.*

Dugué, Daniel. Théorèmes sur les spirales de M. Julia et sur les fonctions absolument monotones. C. R. Acad. Sci. Paris **228**, 40–41 (1949).

I. On sait que, étant donnée une fonction $\varphi(z)$ méromorphe autour du point à l'infini, toute spirale logarithmique $z=\exp(i\delta+(1+ik)\log t)$ (δ réel, k réel, t positif variable) est spirale de Picard ou Julia, quel que soit δ, sauf pour un ensemble $E(\varphi)$ de mesure nulle, de valeurs de k [Valiron, J. Math. Pures Appl. (9) **7**, 113–126 (1928)]. L'auteur précise la nature de $E(\varphi)$ moyennant des hypothèses convenables. Par exemple, si $\varphi(z)$ est holomorphe, il n'y a qu'un nombre fini de valeurs k exceptionnelles de la forme Cn, n entier, quel que soit C. L'auteur indique aussi des cas où $E(\varphi)$ est vide.

II. Si la fonction entière $f(z)=\sum a_n z^n$, $a_n\geqq 0$, est d'ordre ρ et s'il existe une suite d'indices n_p tels que $\lim(n_{p+1}/n_p)=1$, pour lesquels $\sum a_{n_p} z^{n_p}$ est d'ordre inférieur à ρ, la fonction $f(z)$ n'est pas décomposable en un produit de fonctions absolument monotones. De cet énoncé donné sans démonstration, l'auteur déduit qu'une loi de probabilité dérivée

par des combinaisons linéaires de la loi de Poisson, n'est pas décomposable. *G. Valiron* (Paris).

Ghermanescu, Michel. Une inégalité pour les algébroïdes. Bull. Math. Phys. Éc. Polytech. Bucarest 10 (1938–39), 31–33 (1940).

Les fonctions $T(r)$, $N_\nu(r, a_i)$, $S(r)$ ayant la signification habituelle dans la théorie des fonctions algébroïdes d'ordre ν définies par $f_0(z)u^\nu+f_1(z)u^{\nu-1}+\cdots+f_\nu(z)=0$, où les $f_j(z)$ sont holomorphes pour $|z|<R$, H. Cartan a montré [Mathematica, Cluj 7, 5–31 (1933)] que l'on a

$$(q-\nu-1)T(r)<\sum_1^q N_\nu(r, a_i)+S(r), \qquad q>\nu+1,$$

si les $f_j(z)$ sont linéairement indépendantes. Lorsqu'il existe λ relations linéaires à coefficients constants entre les $f_j(z)$, H. Cartan présumait que l'on a

$$(1) \qquad (q-\nu-\lambda-1)T(r)<\sum_1^q N_{\nu-\lambda}(r, a_i)+S(r),$$

et le prouvait lorsque $\lambda=\nu-1$, cas dans lequel on retombe à peu près sur l'inégalité générale donnée par Valiron [C. R. Acad. Sci. Paris 189, 623–625 (1929); voir aussi Bull. Soc. Math. France 59, 17–39 (1931)]:

$$(q-2\nu)T(r)<\sum_1^q N_1(r, a_i)+S(r).$$

S'appuyant sur ses résultats antérieurs [Ann. Sci. École Norm. Sup. (3) 52, 221–268 (1935)] l'auteur présume que le premier membre dans (1) doit être remplacé par $[q-\nu-E(\nu/(\nu-\lambda))]T(r)$. Il montre que l'inégalité que l'on aurait ainsi ne pourrait pas être améliorée. *G. Valiron* (Paris).

Veržbinskiĭ, M. L. On the distribution of the roots of the *L*-transforms of entire transcendental functions. Mat. Sbornik N.S. 22(64), 391–424 (1948). (Russian)

The *L*-transform derives a new function $\phi(x)=\sum_0^\infty a_nP(n)x^n$ from given $f(x)=\sum_0^\infty a_nx^n$. The problem discussed is that of locating the zeros of $\phi(x)$ from information concerning the zeros of $f(x)$ and of $P(x)$. In the classical work of Laguerre [J. Math. Pures Appl. (3) 9, 99–146 (1883); Acta Math. 4, 97–120 (1884)] each part of the hypothesis and also the conclusion was of the form "all zeros real" or "all zeros real and of one sign." In the related work of Pólya and Schur [J. Reine Angew. Math. 144, 89–113 (1914)] and of Benz [Comment. Math. Helv. 7, 243–289 (1935)] the second part of the hypothesis bears on $\sum_0^\infty P(n)x^n$ and not directly on $P(x)$. The results of the present paper include the following fairly straightforward generalisations of the classical results. (a) If $P(x)$ is a polynomial having all its zeros real and negative, if $f(x)$ is of order less than unity and if the convex region D contains the origin and all zeros of $f(x)$ then D contains all zeros of $\phi(x)$. (b) If the $P(x)$ of (a) is replaced by $e^{\gamma x}P(x)$ (γ real) then the zeros of $\phi(x)$ lie within a region $e^{-\gamma}D$ obtained from D by a similitude of centre the origin and ratio $e^{-\gamma}$. (c) If $f(x)$ is a real integral function whose nonreal zeros lie in the strip S: $|\Im(x)|<A$, and if $P(x)$ is a polynomial having real negative zeros then the zeros of $\phi(x)$ also lie within S. (d) If the $P(x)$ of (c) is replaced by $e^{\gamma x}P(x)$ then the zeros of $\phi(x)$ lie within $e^{-\gamma}S$.

Before stating the main results of the paper it seems convenient to summarise the arguments which lead up to them. In the circumstances of (b) or (d) the conclusion may be stated as $|\phi(x)|>0$ if x is outside $e^{-\gamma}D$ ($e^{-\gamma}S$). It can be strengthened to $|\phi(x)|>\lambda>0$ if x is outside a region $D(\lambda)$. But from $|\phi(x)|>\lambda$ and $|x|<R$ it may be possible to deduce $|\phi^*(x)|>0$, where $\phi^*(x)=\sum_0^\infty a_nP^*(n)x^n$ and $e^{\gamma x}P(x)$ is an approximation to $P^*(x)$. Taking $P^*(x)$ an integral function and $e^{\gamma x}P(x)$ as a section of its canonical product, the argument chooses for large R a corresponding $P(x)$ and λ and under suitable conditions conclusions of an asymptotic nature on the location of the zeros of $\phi^*(x)$ are derived. When both $f(x)$ and $P^*(x)$ are of order less than unity, $P^*(x)$ having only real negative zeros, it appears that the results for polynomial $P(x)$ still hold and that the excess of the order of $P^*(x)$ over unity determines the "magnification" in the region containing the zeros. This excess is measured by the definition $l=\lim r^{-1}n(r)$, where $n(r)$ is the number of zeros of $P^*(x)$ in $-r<x<0$. (In cases where this limit does not exist the definition of l is $\lim (\log r)^{-1}\int_0^r t^{-1}dn(t)$.) In particular, if $f(x)$ is real, of order $\omega<2$ and its nonreal zeros lie in the "parabola" $t=\pm(p-\sigma)^q$ $(x=\sigma+it)$ then the nonreal zeros of $\phi^*(x)$ lie in a half plane $\Re(x)<A$ or in a small neighbourhood of the positive real axis provided $1/2(1-q)<\min\{1-l, \omega^{-1}-l\}$. [This special case seems to the reviewer to be the instance of theorem 5 of main interest, for if $0<l<\infty$ the exponent of convergence of the zeros of $P^*(x)$ must be unity while if the exponent of convergence exceeds unity l cannot be finite: cf. E. Lindelöf, Acta Soc. Sci. Fennicae (1) 31, no. 1 (1903), in particular, pp. 5–6.] These results are then generalised by permitting $P^*(x)$ to have a finite number of positive zeros provided they appear in sets of the form $0, 1, \cdots, m, m+\lambda$ $(\lambda<1)$. *A. J. Macintyre* (Aberdeen).

Boas, R. P., Jr. An upper bound for the Gontcharoff constant. Duke Math. J. 15, 953–954 (1948).

Any upper bound for the "Whittaker" constant obtained from solutions of $f'(z)=f(\omega z)$ $(|\omega|=1)$ [Boas, same J. 11, 799 (1944); Macintyre, J. London Math. Soc. 22 (1947), 305–311 (1948); these Rev. 6, 123; 10, 27] is also an upper bound for the "Gontcharoff" constant [Ann. Sci. École Norm. Sup. (3) 47, 1–78 (1930), in particular, p. 17]. This constant is defined in the present paper as the least upper bound of λ for which $F(z)$ regular for $|z|<1$, $F^{(n)}(a_n)=0$, $|na_n|\to\lambda$, $F(z)\not\equiv0$ are possible. The proof depends on the elementary identity $F^{(p)}(z\omega^{-p})=(-)^p\omega^{p(p-1)/2}z^{-p-1}L^{(p)}(z^{-1})$ [note misprints in original], where $F(z)=\sum_0^\infty\omega^{n(n-1)/2}z^n$, $L(z)=\sum_0^\infty\omega^{n(n-1)/2}z^{-n-1}$ and on Widder's theorem [Trans. Amer. Math. Soc. 36, 107–200 (1934), theorem 35, pp. 172–173] connecting zeros of $L^{(p)}(z)$ with zeros of the Laplace transform $f(z)$ of $L(z)$. *A. J. Macintyre* (Aberdeen).

Boas, R. P., Jr. Basic sets of polynomials. II. Duke Math. J. 16, 145–149 (1949).

This continues work of the author [same J. 15, 717–724 (1948); these Rev. 10, 187] on basic polynomial sets. Let $p_n(z)=\sum_k p_{nk}z^k$ be basic, so that to the matrix $P=(p_{nk})$ corresponds a unique row-finite reciprocal $\Pi=(\pi_{kn})$. Let $q_k(z)$ denote the formal power series $\sum_{n=0}^\infty\pi_{nk}z^{-n-1}$. It is shown that if m exists such that $q_m(z)$ is analytic at $z=\infty$, and if the series (*) $\sum_0^\infty d_np_n(z)$ is uniformly summable to zero by a γ-matrix [as defined, e.g., in Dienes, The Taylor Series, Oxford, 1931, p. 397] on a rectifiable Jordan curve C that surrounds the origin and all the singularities of $q_m(z)$, then $d_m=0$. The proof is made to depend on the orthogonality relation $(1/2\pi i)\int_C p_n(z)q_m(z)dz=\delta_{nm}$. The same con-

clusion follows if series (*) is replaced by any subsequence of its partial sums; and more generally, if $q_m(z)$ is allowed to have singularities exterior to C, provided these are isolated and $q_m(z)$ is single-valued exterior to C, then $\sum_{n=0}^{\infty} d_n(\delta_{nm}-R_{nm})=0$ (evaluated by the same γ-matrix), where R_{nm} is the sum of the residues of $p_n(z)q_m(z)$ exterior to C. The usefulness of these results is illustrated by examples, particularly in the case of Appell polynomials.

I. M. Sheffer (State College, Pa.).

Eweida, M. T. The lower bound of the order of a product set of polynomials. Duke Math. J. 16, 119–123 (1949).

The author uses terminology defined by J. M. Whittaker [Interpolatory Function Theory, Cambridge University Press, 1935; Series of Polynomials, Cairo, 1943; these Rev. 8, 454]. Let $\{p_n(z)\}$, $\{q_n(z)\}$ be two simple polynomial sets, and $\{u_n(z)\}$ their product [$u_n(z)=\sum_i p_{ni}q_i(z)$, where $p_n(z)=\sum_i p_{ni}z^i$]; and let ω_1, ω_2, Ω be their respective orders. The following inequalities are shown to hold: if the leading coefficient in $p_n(z)$ and $q_n(z)$ is unity, and if $0\leqq\omega_1<\frac{1}{2}\omega_2$, then $(\frac{1}{2}\omega_2-\omega_1)\leqq\Omega\leqq(\omega_1+2\omega_2)$; if the leading coefficient in $q_n(z)$ is unity, and $\frac{1}{2}\omega_1>\omega_2\geqq 0$, then $(\omega_1-2\omega_2)\leqq\Omega\leqq(\omega_1+2\omega_2)$. [The upper bound $\omega_1+2\omega_2$ is due to M. Nassif, Proc. Math. Phys. Soc. Egypt 3, 43–47 (1946); these Rev. 9, 22. See also Nassif, Amer. J. Math. 71, 40–49 (1949); these Rev. 10, 373.] Examples show that the lower bounds are best possible. *I. M. Sheffer* (State College, Pa.)

Eweida, M. T. On the effectiveness at a point of product and reciprocal sets of polynomials. Proc. London Math. Soc. (2) 51, 81–89 (1949).

According to J. M. Whittaker a basic set of polynomials is said to be effective at $z=0$ if every function $f(z)$ regular at $z=0$ is represented by a series in these polynomials in some circle about $z=0$. (The radius will depend on $f(z)$.) It is here shown that if $\{p_n(z)\}$, $\{q_n(z)\}$ are simple polynomial sets each effective at $z=0$, with leading coefficient in $q_n(z)$ unity, then the product set is also effective at $z=0$. Also, if $\{p_n\}$ is simple and effective at $z=0$, with leading coefficient unity, then the reciprocal set is effective at $z=0$. These results parallel corresponding results for effectiveness in a circle [e.g., Nassif, J. London Math. Soc. 22 (1947), 257–260 (1948); these Rev. 9, 583], but the method of proof is different. Examples are given to show that if the condition (where stated) of leading coefficient unity is removed, the conclusions can be false. *I. M. Sheffer.*

Džrbašyan, M. M. On an extremal problem in the theory of weighted orthogonal polynomials. Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 555–568 (1948). (Russian)

Let $h(z)$ be a bounded positive function in $|z|<1$, remaining constant on each circumference $C_\rho(|z-(1-\rho)|=\rho)$, $0<\rho<1$, and let $h(z)=\lambda(\rho)$ on C_ρ, with $\lambda(\rho)$ nondecreasing in $0<\rho<1$. Denote by A_λ the class of functions $f(z)$, holomorphic in $|z|<1$ and satisfying the conditions

$$\iint_{|z|<1} h(z)\,|f(z)|^2dxdy<\infty,$$

$$\inf_{\{Q\}}\iint_{|z|<1} h(z)\,|f(z)-Q(z)|^2dxdy=0,$$

where $\{Q\}$ is the set of all polynomials. The following three theorems are established. (I) In the class A_λ and subject to the condition $f(\alpha)=1$ (where α is a number in $|\alpha|<1$) the integral (1) $\mu(f)=\iint_{|z|<1}h(z)\,|f(z)|^2dxdy$ is minimized by the (unique) function (2) $f_0(z)=((1-\alpha)/(1-z))^2\phi(z,\bar{\alpha})/\phi(\alpha,\bar{\alpha})$, where

$$(3)\qquad \phi(z,\bar{\alpha})=\int_0^\infty p(t)\exp\left\{-t\left(\frac{1}{1-z}+\frac{1}{1-\bar{\alpha}}\right)\right\}dt,$$

$$1/p(t)=\int_0^1\lambda(\rho)\rho^{-2}e^{-t/\rho}d\rho;$$

and the minimum value is (4) $\mu_0=\inf\mu(f)=\pi|1-\alpha|^4/\phi(\alpha,\bar{\alpha})$. (II) Let $\{p_n(z)\}$ $(n=0,1,\cdots)$ be the uniquely determined set of orthogonal polynomials (p_n of degree n) with leading coefficient positive, defined by

$$\iint_{|z|<1} h(z)p_n(z)\overline{p_m(z)}dxdy=\begin{cases}0, & m\neq n,\\ 1, & m=n.\end{cases}$$

For $|\alpha|<1$, the series $K(z,\bar{\alpha})=\sum_0^\infty\overline{p_n(\alpha)}p_n(z)$ is uniformly and absolutely convergent on every closed set lying in $|z|<1$; and (5) $K(z,\bar{\alpha})=\pi^{-1}\phi(z,\bar{\alpha})(1-z)^{-2}(1-\bar{\alpha})^{-2}$. (III) The functions of class A_λ have the following representation:

$$(6)\qquad f(z)=\frac{1}{\pi(1-z)^2}\iint_{|w|<1} h(w)F(w)\frac{\phi(z,\bar{w})}{(1-\bar{w})^2}dudv$$

$(w=u+iv)$, where $F(w)$ is an arbitrary function satisfying the condition (7) $\iint_{|w|<1}h(w)\,|F(w)|^2dudv<\infty$.

The proof of (I) is made to depend on an auxiliary extremal problem. Let $0<x_1<1$, $x_1x_2=1$, and let $C_\rho(x_1)$ be the circle tangent to C_ρ at the point $1-2\rho$ and such that x_1, x_2 are symmetric to $C_\rho(x_1)$. Define the weight function $h(z,x_1)$ to have the constant value $\lambda(\rho)$ on $C_\rho(x_1)$. Then a unique function $f(z)=f_0(z,x_1)$ is found, holomorphic in $|z|<1$, with $f(\alpha)=1(|\alpha|<1)$, that minimizes the integral $\iint_{|z|<1}h(z,x_1)\,|f(z)|^2dxdy$. It is then shown that $\lim_{x_1\to 1}f_0(z,x_1)=f_0(z)$ (defined by (2)), and that this is the unique solution of the original extremal problem. Theorems II and III are then shown to be consequences. The relation between $F(w)$ and $f(z)$ of (6) is this: if a_k is defined by $a_k=\iint_{|w|<1}h(w)F(w)\overline{p_k(w)}dudv$, then $f(z)=\sum_0^\infty a_kp_k(z)$.

The following particular cases of the above theory are briefly considered: $\lambda(\rho)=1$; $\lambda(\rho)=0$, $0<\rho<r$; $=1$, $r\leqq\rho<1$; $\lambda(\rho)=\exp(1-1/\rho)$. Misprints: page 556, theorem II, the series for $K(z,\bar{\alpha})$ should begin with $n=0$; page 559, equation (*), replace $F(n,h)$ by $F(nh)$; page 567, line 4, insert the factor $F(w)$ in the integrand. *I. M. Sheffer.*

Dalzell, D. P. On the theory of functions associated with a canonical Fuchsian group. Proc. London Math. Soc. (2) 51, 90–113 (1949).

A canonical group is defined as a group of linear substitutions consisting wholly of hyperbolic transformations of the interior of the unit circle onto itself and having a fundamental region wholly within a circle about the origin of radius less than unity. The aim of the paper is to simplify the classical Poincaré theory of automorphic functions invariant under such a group. This is achieved by modelling the theory on the Weierstrass theory of elliptic functions. Functions closely analogous to the functions $\wp$, ζ, σ of Weierstrass are defined with respect to a given canonical group and their relations to the functions employed by Poincaré are exhibited. *Z. Nehari* (St. Louis, Mo.).

Dalzell, D. P. Algebraic relations between theta-Fuchsian functions. Proc. London Math. Soc. (2) 51, 114–131 (1949).

The theta-Fuchsian functions $\theta(z; m)$ of rank m belonging to a discrete group of linear substitutions $\{T_\nu\}$ (functions fundamental in the Poincaré theory of automorphic functions) have the transformation property (*) $\theta(T_\nu z; m)[T_\nu' z]^m = \theta(z; m)$. Property (*) implies that a homogeneous polynomial of θ-functions of rank unity is a θ-function of rank equal to that of the polynomial. The author shows that in the case in which $\{T_\nu\}$ is a canonical group [see the preceding review] which is not hyperelliptic and $\theta(z; m)$ is regular in $|z|<1$, the reverse is also true, i.e., a θ-function of rank m is expressible as a homogeneous polynomial of θ-functions of rank unity associated with the same canonical group. *Z. Nehari* (St. Louis, Mo.).

Petersson, Hans. Über die Berechnung der Skalarprodukte ganzer Modulformen. Comment. Math. Helv. 22, 168–199 (1949).

Let Γ be a congruence subgroup of finite index μ in the full modular group. The function $f(\tau)$ is said to belong to $\{\Gamma, -r, v\}$ if and only if (1) f is regular in $\mathfrak{J}(\tau)>0$, (2) $f(L\tau)=v(L)(\gamma\tau+\delta)^r f(\tau)$ for every $L=\binom{\alpha\ \beta}{\gamma\ \delta}\subset\Gamma$, where $|v(L)|=1$ and $r>0$, (3) $f(\tau)$ has at most poles (measured in the local uniformising variable) at the parabolic vertices of Γ. If N is the smallest positive integer such that $\binom{1\ N}{0\ 1}\subset\Gamma$, define κ by $\exp 2\pi i\kappa = v(\binom{1\ N}{0\ 1})$, $0\leqq\kappa<1$. Then f has a Fourier series $f(\tau)=\sum_{n=0}^{\infty} b_n \exp 2\pi i(n+\kappa)\tau/N$.

In previous papers [cf., e.g., Math. Ann. 117, 453–537 (1940); these Rev. 2, 87] the author has introduced the scalar product (f, g) of two entire modular forms belonging to $\{\Gamma, -r, v\}$:

$$(f, g; \Gamma)=(f, g)=\int\int_{\mathfrak{F}} f(\tau)\overline{g(\tau)}y^{r-2}dxdy, \qquad \tau=x+iy,$$

where $\mathfrak{F}$ is any fundamental region of Γ. In the present paper he establishes a connection between the scalar product of f, g and a certain Dirichlet series built up out of their Fourier coefficients, as follows. Theorem 6: $f(\tau)$, $g(\tau)$ belong to $\{\Gamma, -r, v\}$, $r>0$, and $f(\tau)g(\tau)$ is a cusp-form, i.e., fg vanishes at all parabolic vertices of Γ. Let b_n, c_n $(n\geqq 0)$ be the Fourier coefficients of f, g respectively, and construct the Dirichlet series $D(s; f, g; \Gamma)=\sum_{n+\kappa>0} b_n\bar{c}_n(n+\kappa)^{-s}$. Then the series converges absolutely for $\sigma=\Re(s)>2r$, and $D(s)$ can be continued analytically to the half-plane $\sigma>r-\frac{1}{2}$, and is regular there with the possible exception of the point $s=r$, where it may have a simple pole. Moreover,

$$\text{(1)} \qquad \operatorname*{residue}_{s=r} D(s; f, g; \Gamma)=\frac{(4\pi)^{r-1}}{N^r\Gamma(r)}\frac{12}{\mu}(f, g; \Gamma).$$

When $0<r<1$, the above result is essentially true for forms f, g such that fg is not necessarily a cusp-form. With $0<r<1$ and f, g arbitrary entire modular forms belonging to $\{\Gamma, -r, v\}$, the conclusions of theorem 6 hold with the modification that $D(s)$ is analytically continuable to the half-plane $\sigma>\max(r-\frac{1}{2}, 2r-1, 0)$.

The author then calculates, by use of (1), the values of scalar products of various theta functions of dimension $-\frac{1}{2}$ and $-\frac{3}{2}$, e.g.,

$$\vartheta_3(\tau)=\sum_{-\infty}^{\infty}\exp \pi i\tau n^2, \quad \eta(\tau)=\exp \pi i\tau/12\prod_{1}^{\infty}(1-e^{2\pi i n\tau}),$$

and some of higher level (Stufe),

$$\vartheta_\lambda(\tau, h, N)=\sum_{m\equiv h(N)} m^\lambda \exp \pi i\tau m^2/N, \qquad \lambda=0, 1.$$

A more interesting example is promised in a future work, of which one case is

$$\zeta(3)=\frac{\pi^3}{7}(\vartheta_3^5, \vartheta_0^4\vartheta_2^4);$$

ϑ_3 has been defined above,

$$\vartheta_0(\tau)=\sum_{-\infty}^{\infty}(-1)^n \exp \pi i n^2\tau, \quad \vartheta_2(\tau)=\sum_{-\infty}^{\infty}\exp \pi i(n+\tfrac{1}{2})^2\tau.$$

The scalar product appears in this formula as the analogue of the Bernoulli numbers in the classical expression for $\zeta(2m)$. *J. Lehner* (Philadelphia, Pa.).

Roure, Henri. Sur une classe nouvelle de fonctions. III. Ann. Fac. Sci. Univ. Toulouse (4) 9 (1945), 65–72 (1948).

For the first two parts cf. the same Ann. (4) 6, 15–31 (1943); 7, 99–122 (1945); these Rev. 7, 380.

Matsumoto, Toshizô. On certain hypercomplex numbers. Jap. J. Math. 19, 441–482 (1947).

L. Sobrero [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 77–82, 135–140, 479–483 (1934)] has employed hypercomplex numbers $z=a_0+a_1j+a_2j^2+a_3j^3$, where the a_i are real and the unit j satisfies $(1+j^2)^2=0$, in discussing the biharmonic equation $\Delta\Delta\psi=0$, which occurs in the theory of plane elasticity. Following this idea, Tunezo Sato is said to have considered hypercomplex numbers $z=a_0+a_1j+a_2j^2+a_3j^3$, where the a_i are real and the unit j satisfies $e_1+e_2j^2+j^4=0$ $(e_1>0, e_2>0)$, in discussing the equation $e_1\partial^4\psi/\partial x^4+e_2\partial^4\psi/\partial x^2\partial y^2+\partial^4\psi/\partial y^4=0$, which occurs in the theory of aeolotropic plates. The present author is concerned with the function theory of the hypercomplex numbers in question. The hypercomplex numbers dealt with are $z=\alpha+\beta\tilde{\omega}$, where α and β are ordinary complex numbers and $\tilde{\omega}$ is a unit; $z=\alpha+\beta\tilde{\omega}$ and $z'=\alpha'+\beta'\tilde{\omega}$ are defined to be equal if and only if $\alpha=\alpha'$ and $\beta=\beta'$. Sum, difference, and product are defined as follows:

$$z\pm z'=(\alpha\pm\alpha')+(\beta\pm\beta')\tilde{\omega},$$
$$zz'=\alpha\alpha'+(\alpha\beta'+\alpha'\beta+\beta\beta'\omega)\tilde{\omega},$$

where ω is a certain real constant. The two cases $\omega\neq 0$ and $\omega=0$ are treated separately. In each case the notion of an analytic hypercomplex-valued function $f(z)$ of the hypercomplex variable $z=\alpha+\beta\tilde{\omega}$ is introduced, and many of the familiar theorems for ordinary analytic functions: convergence of power series, Taylor's and Laurent's expansions, Cauchy's integral theorem and Cauchy's formula, etc., are shown to remain valid. The connection with the hypercomplex number systems of order four over the real field employed by Sobrero and Sato is obtained by considering the roots of the equation $e_1+e_2j^2+j^4=0$, $e_1>0$, $e_2>0$, $e_2^2-4e_1\geqq 0$. The connection with partial differential equations follows in the usual way by the consideration of the corresponding Cauchy-Riemann equations. *J. B. Diaz.*

Takasu, Tsurusaburo. Theorie der Funktionen einer allgemeinen bikomplexen Veränderlichen. I. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. 32, 1–55 (1945).

The author develops the theory of functions of a bicomplex variable $\zeta=x+jy+j'(u+jz)$, where $j^2=\mu+\nu j$, $j'^2=\mu'+\nu'j'$, (x, y, u, z) are real variables, and (μ, ν, μ', ν') are real constants. Various special cases have been studied

by M. Futagawa [Tôhoku Math. J. 29, 175–222 (1928)], F. Ringleb [Rend. Circ. Mat. Palermo 57, 311–340 (1933)], G. Scorza Dragoni [Accad. Italia Mem. Cl. Sci. Fis. Mat. Nat. 5, 597–665 (1934)] and N. Spampinato [Ann. Mat. Pura Appl. (4) 14, 305–325 (1936)]. The first part of the paper discusses the algebra of such bicomplex numbers. An appropriate trigonometry is defined in each of the xy, xu, xz coordinate planes so that a bicomplex variable may be written in polar form. The various algebraic manipulations are obtained in polar form. The bicomplex variables can be represented as points on a certain four-dimensional quadric embedded in five-dimensional space by a process analogous to stereographic projection.

The topology of the bicomplex four-dimensional space is discussed. Various theorems on continuous functions are obtained. The analogues of well-known theorems on infinite series of functions and the radius of convergence of a power series are obtained. The analogue of the Cauchy-Riemann equations consists of sixteen conditions, and that of the Laplace equation consists of four conditions. In this connection, the author discusses the analogue of the Looman-Menchoff theorem. A power series is always holomorphic. The Taylor series with a remainder is obtained. The analogue of conformal mapping is discussed. *J. De Cicco.*

Theory of Series

***Knopp, Konrad. Theorie und Anwendung der Unendlichen Reihen.** 4th ed. Springer-Verlag, Berlin and Heidelberg, 1947. xii+583 pp. 39.60 DM.

The earlier editions of this well-known text and reference book appeared in 1922 [474 pp.], 1924 [527 pp.] and 1931 [582 pp.]. A translation of edition 2 by R. C. Young appeared in 1928 [Theory and Application of Infinite Series, Blackie, London, 571 pp.]. The new edition is, chapter by chapter and section by section, identical with the second except that the new edition introduces a few additional short remarks, references, and simplifications. The major simplification is the use of the idea of Karamata [Math. Z. 32, 319–320 (1930)] for proofs of Tauberian theorems for power-series summability. *R. P. Agnew.*

Walsh, C. E. Some series for π. Edinburgh Math. Notes 37, 20–21 (1949).

Macintyre, A. J. Euler's limit for e^x and the exponential series. Edinburgh Math. Notes 37, 26–28 (1949).

Bradley, F. W., and Edrei, A. On the ratios of one term to the remainders in a convergent series of positive terms. J. London Math. Soc. 24, 60–64 (1949).

Let $p_1+p_2+\cdots$ be a convergent series of positive terms. The question treated is that of existence, finite or infinite, of numbers $L_0, L_{-1}, L_1, \cdots$ defined by

$$\lim_{n\to\infty}\frac{p_{n+k}}{p_n+p_{n+1}+p_{n+2}+\cdots}=L_k.$$

If L_0 exists, then L_k exists for each k and $L_k=L_0(1-L_0)^k$, where $(1-L_0)^k$ is to be interpreted as $+\infty$ if $L_0=1$ and $k<0$. If, for some $k<0$, L_k exists and is finite, then L_0 exists and is the unique root in $0\leqq x<1$ of the equation $x(1-x)^k=L_k$. If, for some $k>0$, L_k exists and $L_k\neq 0$, then L_0 exists and is one of the two roots in $0<x<1$ of the equation $x(1-x)^k=L_k$. *R. P. Agnew* (Ithaca, N.Y.).

Edrei, Albert. Sur des suites de nombres liées à la théorie des fractions continues. Bull. Sci. Math. (2) 72, 45–64 (1948).

A sequence of positive numbers $\{c_n\}$ is called a positive proper chain sequence if there exists a sequence $\{g_n\}$ satisfying $0<g_n<1$ such that $c_1=g_1$, $c_n=g_n(1-g_{n-1})$, $n\geqq 2$. It is known that the continued fraction $1/(1+K(-c_n/1))$ is equivalent to the series

$$1+\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}g_k/(1-g_k)\right)$$

and that this relation can be employed in deriving convergence criteria for continued fractions of the form

$$(1)\qquad \frac{1}{1+K(-c_nx_n/1)},$$

where c_n is a chain sequence and $|x_n|\leqq 1$. This paper contains a systematic development of certain properties of chain sequences. The results are used to prove convergence criteria for (1). None of these criteria are new. The improvement over Van Vleck's theorem contained in theorem 1 was first made by Paydon and Wall [Duke Math. J. 9, 360–372 (1942); these Rev. 3, 297]. *W. J. Thron.*

Shah, S. M. A note on quasi-monotone series. Math. Student 15 (1947), 19–24 (1948).

The essential results of this paper are contained in a paper by the reviewer [Amer. J. Math. 70, 203–206 (1948); these Rev. 9, 278], as stated by the author in a postscript. *O. Szász* (Cincinnati, Ohio).

Zygmund, A. Two notes on the summability of infinite series. Colloquium Math. 1, 225–229 (1948).

The first note gives proofs of Hardy's O-Tauberian theorem for C_1 summability (if $\sum u_n$ is summable C_1 and $nu_n=O(1)$, then $\sum u_n$ converges) and Landau's O_L-Tauberian theorem with the unilateral condition $nu_n>-K$. The proofs employ classic ideas [pertinent references may be found in Agnew, Ann. of Math. (2) 42, 293–308 (1941); these Rev. 2, 191]. They are very short and easy to comprehend.

The second note involves the transformation

$$(1)\qquad \sigma(t)=u_0+\sum_{k=1}^{\infty}(kt)^{-2}\sin^2(kt)u_k$$

of Riemann summability R_2. When $\sum u_k$ is a series with bounded partial sums s_k, this can be put in the form $\sigma(t)=\sum a_k(t)s_k$, where

$$(2)\qquad a_k(t)=\left(\frac{\sin kt}{kt}\right)^2-\left(\frac{\sin (k+1)t}{(k+1)t}\right)^2.$$

It is well known that $(R, 2)$ is regular and hence that the number λ defined by (3) $\limsup_{t\to 0}\sum_{k=0}^{\infty}|a_k(t)|=\lambda$ is finite. Despite the fact that the value of λ is of importance in the theory of summability, its value has not previously been determined, the best previous estimate being $\lambda\leqq 1+2/\pi^2$. [See Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, v. 2, 2d ed., Cambridge University Press, 1926, pp. 224–225, and references given there.] The author shows that the sum in (3) has a limit λ as $t\to 0$ and that $\lambda=\frac{1}{2}(e^2-5)=1.1945<1.2026=1+2/\pi^2$. The constant λ is seen to be the total variation over $0<x<\infty$

of $\sin^2 x/x^2$, and this total variation is equal to the value of a convergent series which is evaluated by use of complex integrals. The same method is said to apply to the Riemann methods R_{2k} obtained by replacing the exponents 2 in (1) and (2) by $2k$, but the limits are not computed.

R. P. Agnew (Ithaca, N. Y.).

Basu, S. K. On the total relative strength of the Riesz and Cesàro methods. J. London Math. Soc. **24**, 51–59 (1949).

Let C and R denote respectively the Cesàro method (C, r) and the Riesz method (R, n, r) for summability of series, and let $A\sum u_n = L$ abbreviate the statement that $\sum u_n$ is summable A to L. A well known equivalence theorem says that, when $r > -1$, $C\sum u_n = R\sum u_n$ whenever one of $C\sum u_n$ and $R\sum u_n$ is finite. The following theorems apply to series $\sum u_n$ of real terms. If r is a positive integer, then $C\sum u_n = +\infty$ implies $R\sum u_n = +\infty$. If r is positive and not an integer, then $C\sum u_n = +\infty$ does not imply $R\sum u_n = +\infty$. If $0 \leqq r \leqq 1$, then $R\sum u_n = +\infty$ implies $C\sum u_n = +\infty$. If $r > 1$, then $R\sum u_n = +\infty$ does not imply $C\sum u_n = +\infty$.

R. P. Agnew.

Basu, S. K. On the total relative strength of the Riesz and Hölder methods. Bull. Calcutta Math. Soc. **40**, 153–162 (1948).

In addition to the notation of the preceding review, let H denote the Hölder method (H, r). If $r > 0$ but $r \neq 1$, then $H\sum u_n = +\infty$ does not imply $R\sum u_n = +\infty$. If $r > 1$, then $R\sum u_n = +\infty$ does not imply $H\sum u_n = +\infty$. If $0 < r < 1$, the question whether $R\sum u_n = +\infty$ implies $H\sum u_n = +\infty$ is left undecided.

R. P. Agnew (Ithaca, N. Y.).

Kuttner, B. The relation between different types of Abel summability. Proc. Cambridge Philos. Soc. **45**, 186–193 (1949).

A series $\sum a_n$ is summable to L by the generalized Abel method (or Dirichlet series method) (A, μ_n), and we write $(A, \mu_n)\sum a_n = L$, if the series $\sum a_n \exp(-\mu_n s)$ converges when $s > 0$ to a function $f(s)$ such that $f(s) \to L$ as $s \to 0+$. A function $\varphi(x)$ which is defined for $x > 0$, increasing for $x > x_0$, and such that $\varphi(x) \to \infty$ as $x \to \infty$, has property D if the following is true. If $0 < \mu_1 < \mu_2 < \cdots$, $\mu_n \to \infty$, $(A, \mu_n)\sum a_n = L$, and if $g(s) \equiv (A, \mu_n)\sum a_n \exp\{-\varphi(\mu_n)s\}$ exists for each $s > 0$, then $g(x) \to L$ as $s \to 0+$. The known facts that $\log x$ and, when $0 < r \leqq 1$, x^r have property D, together with known facts about Riesz summability, make it natural to expect that each reasonably regular function increasing no more rapidly than x would have property D. However the function $x - \log x$ fails to have property D. By use of Laplace integrals, it is shown that $\varphi(x)$ has property D if there exist positive constants A, δ, and K such that

$$\sum_{k=0}^{\infty} \left| \frac{d^k}{dx^k} \exp(-s\varphi(x)) \right| (x-A)^k/k! < K$$

when $x > A$ and $0 < s < \delta$. It is not known whether each $\varphi(x)$ having property D must satisfy this condition.

R. P. Agnew (Ithaca, N. Y.).

Cooke, Richard G., and Barnett, A. Mary. The "right" value for the generalized limit of a bounded divergent sequence. J. London Math. Soc. **23**, 211–221 (1948).

Confining attention to bounded sequences, the authors say that a series $\sum c_n$, and its sequence s_n of partial sums, have the "right" value L if they are summable to L by the power series method, that is, if $\sum c_n z^n \to L$ as $z \to 1-$. [The reviewer does not agree with the apparent implication of the term "right" value.] It is shown that a given bounded sequence having a finite set of limit points must be evaluable by a given regular matrix method T if some suitably related sequences of zeros and ones are so evaluable. Some known theorems about evaluations of sequences of zeros and ones are phrased in terms of "right" values. Conditions are given which imply that a matrix method A includes the Cesàro method C_r of a given positive order r and hence evaluates almost all sequences of zeros and ones to $\frac{1}{2}$.

R. P. Agnew (Ithaca, N. Y.).

Wing, G. Milton. Summability with a governor of integral order. Bull. Amer. Math. Soc. **55**, 146–155 (1949).

Corresponding to an arbitrary series (1) $\sum_{n=0}^{\infty} a_n$ of complex numbers, let $p_n(0) = |a_n|$ and $p_n(k) = \sum_{j=0}^{n} p_j(k-1)$ $(n = 0, 1, \cdots; k = 1, 2, \cdots)$. For every positive integer k, the author defines the series (1) to be summable (G, k) provided the Nörlund method $(N, p_n(k))$ is regular and sums the series; appropriate special provision is made for the case where some of the quantities $p_n(k)$ are zero. For every positive integer k, the relation $(G, k) \subset (G, k+1)$ holds, and the inclusion sign cannot be reversed; the same is true for the relation $(C, k) \subset (G, k)$, where (C, k) denotes the Cesàro method of order k. The author gives sufficient conditions for the series (1), assumed to be summable (C, k), to be summable (G, r), where r is a positive integer less than k.

G. Piranian (Ann Arbor, Mich.).

Bruynes, H., and Raisbeck, G. A method of analytic continuation suggested by heuristic principles. Bull. Amer. Math. Soc. **55**, 193–197 (1949).

Let $f(z)$ be analytic over the set $|z| < 1$. Elementary estimates suggest that, at least when $|z| < 1$, the function $\sigma_n(z)$ defined by

$$\sigma_n(z) = \sum_{m=0}^{n} \binom{n}{m} f^{(m)}(0)(z/n)^m$$

should be near $f(z)$ when n is large. Let E be the interior of the outer loop of the curve whose equation in polar coordinates is $\cos\theta = r(1 - \log r)$ If F is a closed subset of E, and $f(z) = 1/(1-z)$, then $\sigma_n(z) \to f(z)$ uniformly over F. For other functions $f(z)$, this result and the Cauchy integral formula show that the region of convergence of $\sigma_n(z)$ is the intersection of regions, similar to F, determined by the singular points of $f(z)$. The method is that by which the classic Borel polygon of summability is usually obtained.

R. P. Agnew (Ithaca, N. Y.).

Rajagopal, C. T. On an absolute constant in the theory of Tauberian series. Proc. Indian Acad. Sci., Sect. A. **28**, 537–544 (1948).

Let λ_n be an increasing sequence of positive numbers for which $\lambda_n \to \infty$ and $\lambda_n - \lambda_{n-1}$ is bounded. Let $\sum a_n$ be a series of complex terms satisfying the Tauberian condition

$$\limsup_{n\to\infty} \lambda_n |a_n| / (\lambda_n - \lambda_{n-1}) = K < \infty.$$

Let L and L_D denote, respectively, the set of limit points of the sequence s_n of partial sums of $\sum a_n$ and the set of limit points, for $s \to 0$, of the Dirichlet series transform $F(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s}$, $s > 0$, of $\sum a_n$. Let

$$\rho_1 = \gamma + \log\log 2 + 2\int_{\log 2}^{\infty} x^{-1} e^{-x} dx,$$

where γ is Euler's constant. Then to each s' in L corre-

sponds a z'' in L_D such that $|z'-z''|\leqq\rho_1K$. The constant ρ_1 cannot be replaced by a smaller one. To each z'' in L_D corresponds a z' in L such that $|z'-z''|\leqq\rho_1K$. This generalizes theorems, with the same constant ρ_1, of the reviewer [Duke Math. J. 12, 27–36 (1945); these Rev. 7, 12] for the case when $\lambda_n=n$ and the Dirichlet series transform is the Abel power-series transform. For further results and references see a more recent paper [Ann. of Math. (2) 50, 110–117 (1949); these Rev. 10, 291]. *R. P. Agnew.*

Avakumović, Vojislav G. Contribution à la théorie des intégrales de Laplace. Acad. Serbe Sci. Publ. Inst. Math. 2, 91–107 (1948). (French. Serbian summary)

[The paper is dated 1941.] The author continues his researches on the integral

$$I(s)=s\int_0^\infty e^{-su}A(u)du \qquad (\sigma=\Re s>0),$$

on the general hypothesis that $I(s)$ is bounded in a convex domain D_k having contact of order $k-1$ $(1<k<\infty)$ with the imaginary axis at $s=0$ [cf. Math. Z. 46, 650–664 (1940); 47, 141–152 (1940); these Rev. 2, 191; 3, 232]. He writes (with a fixed $\lambda>0$) $Q(t)=\lim_{\sigma\to+0} I(\sigma+|t|^k+i\lambda t)$, $H(y)=\pi^{-1}\int_{-\infty}^{\infty}t^{-1}\sin yt\,Q(t)dt$ (so that $H(y)$ is the Dirichlet integral of the boundary values of $I(s)$ for a certain D_k), and aims at obtaining inferences from $H(y)$ to $A(u)$ subject to the Tauberian condition

$$\text{(K-o)}\quad \liminf_{u\to\infty}\ \min_{u\leqq u'\leqq U} u^\beta\{A(u')-A(u)\}=-\omega(\epsilon)\to0 \qquad (\epsilon\to+0),$$

where $U=u+\epsilon u^{1/k}$, $0\leqq\beta<(k-1)/k$. The principal result in this direction (a refinement of an earlier one) is that $H(y)=Q(0)+o(y^{-\beta})\to A(u)=Q(0)+o(u^{-\beta})$ $(y,u\to\infty)$, whenever (K-o) is satisfied. The main part of the paper is devoted to an intermediate class of theorem in which (K-o) is not assumed and the conclusions relate to certain weighted averages of $A(u)$ (with "peaked" weighting functions of rather elaborate structure). The passage to $A(u)$ itself, subject to (K-o), is indicated without full details; it is stated that the steps omitted follow familiar Tauberian lines.

A. E. Ingham (Cambridge, England).

Fourier Series and Generalizations, Integral Transforms

Hardy, G. H., and Littlewood, J. E. A new proof of a theorem on rearrangements. J. London Math. Soc. 23, 163–168 (1948).

The paper gives a new and short proof of the following inequality which is basic in the theory of rearrangement of Fourier coefficients. Suppose that $c_0, c_1, \cdots, c_n$ are non-negative numbers, that exactly $\nu+1$ are positive, and that $c_m\leqq c_0$. Suppose that q is an even positive integer. Denote by $\{c_m^*\}$ the sequence $\{c_m\}$ arranged in descending order of magnitude. Writing

$$f=c_0+2\sum_1^n c_m\cos m\theta,\quad f^*=c_0+2\sum_1^\nu c_m^*\cos m\theta,$$

one has

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|f|^q d\theta\leqq(q-1)(2\pi)^{-1}\int_{-\pi}^{\pi}|f^*|^q d\theta.$$

R. Salem (Cambridge, Mass.).

Turán, Paul. On the strong summability of Fourier series. J. Indian Math. Soc. (N.S.) 12, 8–12 (1948).

It is familiar in the theory of Fourier series that, if $f(x)$ is a continuous function of period 2π, then

$$n^{-1}\sum_{\nu=0}^{n}|s_\nu(x)-f(x)|^k\to0$$

for any given positive k, the $s_\nu(x)$ being the partial sums of the Fourier series of $f(x)$. It is shown, by means of an example of the "Fejér type," that this is no longer true when k is replaced by an arbitrary sequence of positive numbers k_n increasing to infinity. *W. W. Rogosinski.*

Timan, A. F. Certain asymptotic estimates for the polynomials of N. I. Ahiezer and B. M. Levitan. Doklady Akad. Nauk SSSR (N.S.) 64, 175–178 (1949). (Russian)

Let $0<\theta_n\leqq1$ for $n=1, 2, \cdots$ and let

$$G_n(x)=\pi^{-1}x^{-2}\{\cos((1-\theta_n)x/\theta_n)-\cos(x/\theta_n)\}.$$

With every $f(x)$, $-\infty<x<\infty$, such that $f(x)/(1+x^2)\epsilon L$, we associate the operators

$$\sigma_n(f;\theta_n,x)=\int_0^\infty\{f(x+t/(n\theta_n))+f(x-t/(n\theta_n))\}G_n(t)dt.$$

If f is periodic and $\theta_n=1$ or $\theta_n=1/n$, the σ_n reduce respectively to the Fejér means or the partial sums of the Fourier series of f. Let C be the class of all functions continuous in $(-\infty,+\infty)$ and $\bar{C}$ the class of all continuous functions of period 2π. Let L_n and $\bar{L}_n$, respectively, denote the norms of $\sigma_n(f;\theta_n,x)$ and $\sigma_n(f;\theta_n,x)$ in C and $\bar{C}$. It is shown that $\bar{L}_n=L_n+o(1)=4\pi^{-2}\log\theta_n^{-1}+o(1)$ if $\theta_n\geqq1/n$, and that $\bar{L}_n=4\pi^{-2}\log n+o(1)$, $L_n=4\pi^{-2}\log\theta_n^{-1}+o(1)$ for $\theta_n\leqq1/n$. A necessary and sufficient condition that $\lim\sigma_n(f;\theta_n,x)=f(x)$ for every $f\epsilon\bar{C}$ is that $\liminf\theta_n>0$. *A. Zygmund.*

Timan, A. F., and Ganzburg, M. M. On the convergence of certain processes for the summation of Fourier series. Doklady Akad. Nauk SSSR (N.S.) 63, 619–622 (1948). (Russian)

Let C be the space of all functions $f(x)$ continuous and of period 2π, and let $S_n(f,x)$ be the nth partial sum of the Fourier series of f. Given p sequences of numbers $\{\alpha_n^1\}, \{\alpha_n^2\}, \cdots, \{\alpha_n^k\}$ satisfying the condition (*) $\alpha_n^k=O(1/n)$ for all k, the authors consider the expression

$$(\ddagger)\qquad \omega_n(\alpha_n^1,\cdots,\alpha_n^p;x,f)=p^{-1}\sum_{k=1}^{p}S_n(f,x+\alpha_n^k)$$

and show that a necessary and sufficient condition for (‡) to tend to $f(x)$ for all x is that

$$\sum_{i,j=1}^{p}\cos(n+\tfrac12)(\alpha_n^i-\alpha_n^j)=O(\log^{-2}n).$$

More generally it is shown that the norm of the operator (‡) in the space C is

$$4p^{-1}\pi^{-2}\log n\left\{\sum_{i,j=1}^{p}\cos(n+\tfrac12)(\alpha_n^i-\alpha_n^j)\right\}+O(1).$$

Let $\{\alpha_n\}$ be a sequence tending to 0 and $r\geqq0$ a fixed integer. For the sums $\omega_n^{(r)}(\alpha_n;f,x)=2^{-r}\sum_{i=0}^{r}\binom{r}{i}S_n(f,x+i\alpha_n)$ the authors show that they tend to $f(x)$, for all $f\epsilon C$, if and only if there is a function $m=m(n)$ $(n=1,2,\cdots)$ taking only a finite number of (integral) values and such that $\alpha_n=2m\pi/(2n+1)+O(n^{-1}\log^{-1/r}n)$. These theorems gener-

alize some of the well-known results of Rogosinski [Math. Ann. 95, 110–134 (1925); Math. Z. 25, 132–149 (1926)].

A. Zygmund (Chicago, Ill.).

Ganzburg, I. M. On a method of approximation of continuous functions by trigonometric sums. Doklady Akad. Nauk SSSR (N.S.) 64, 13–16 (1949). (Russian)

Let C be the class of all continuous functions of period 2π, and let $S_n(f, x)$ denote the partial sums of the Fourier series of any $f\epsilon C$. Given any real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ we consider the expressions

$$\omega_n(\alpha_n, \beta_n; f, x) = \tfrac{1}{3}\{S_n(f, x) + S_n(f, x+\alpha_n) + S_n(f, x+\beta_n)\}$$

(thus we may assume that $|\alpha_n|, |\beta_n| \leqq \pi$ for all n). It is shown that a necessary and sufficient condition that $\omega_n(\alpha_n, \beta_n; f, x)$ tend uniformly to every $f\epsilon C$ is that

$$\alpha_n = \frac{4p\pi}{3(2n+1)} + O(1/(n \log n)),$$

$$\beta_n = \frac{4q\pi}{3(2n+1)} + O(1/(n \log n)),$$

where $p = p(n)$ and $q = q(n)$ are integers taking only a finite number of values, of the form $3r-1$ and $3s+1$, respectively. [For a result of similar type see the preceding review.]

A. Zygmund (Chicago, Ill.).

Zamansky, Marc. Sur l'approximation des fonctions continues périodiques. C. R. Acad. Sci. Paris 228, 460–461 (1949).

The following are some of the results announced by the author. (1) Let σ_n be the $(C, 1)$ means of the Fourier series of f, and let $\bar{f}$ be the function conjugate to f. Then a necessary and sufficient condition that $\bar{f}\epsilon\mathrm{Lip}\,1$ is that $\sigma_n - f = O(1/n)$ uniformly in x. [The necessity of the condition had been known; see Alexits, Mat. Fiz. Lapok 48, 410–422 (1941); these Rev. 8, 261]. (2) Suppose that $f(x)\epsilon\mathrm{Lip}\,\alpha$, $0<\alpha<1$, is of period 2π and mean value zero. Then a necessary and sufficient condition for the fractional (Weyl) integral of f of order $1-\alpha$ to belong to Lip 1 is that

$$\int_{t=\epsilon}^{\infty}\int_{u=0}^{\infty}\frac{\bar{f}(x+t-u)+\bar{f}(x-t-u)-2\bar{f}(x-u)}{t^2u^\alpha}\,du\,dt$$

is bounded uniformly in x and $\epsilon>0$. (3) A necessary and sufficient condition that $\bar{f}\epsilon\mathrm{Lip}\,1$ is that

$$\int_\epsilon^\infty t^{-2}\{f(x+t)+f(x-t)-2f(x)\}\,dt$$

is uniformly bounded in x and $\epsilon>0$. *A. Zygmund.*

Minakshisundaram, S. Notes on Fourier expansions. III. Fourier Stieltjes series. Amer. J. Math. 71, 60–66 (1949).

[For parts I and II see J. London Math. Soc. 20, 148–153 (1945); 21 (1946), 264–267 (1947); these Rev. 8, 150; 9, 141.] Let $F(e)$ be an additive function of bounded variation defined on the Borel sets of the k-dimensional interval $0 \leqq x_1, x_2, \cdots, x_k \leqq 2\pi$, and let

$$(*) \qquad \sum C_{n_1,\cdots,n_k} \exp i(n_1x_1+\cdots+n_kx_k)$$

be the Fourier-Stieltjes series of dF, so that

$$C_{n_1\cdots n_k} = (2\pi)^{-k}\int_0^{2\pi}\cdots\int_0^{2\pi}\exp\{-i(n_1x_1+\cdots+n_kx_k)\}\,dF(e).$$

It is shown that (1) at almost every point, series (*) is summable $(\nu^2, k-\frac{1}{2}+\epsilon)$, $\epsilon>0$. Moreover, (2) at every point where the symmetric derivative of $F(e)$ (that is, the density of $F(e)$ with respect to k-dimensional spheres with center at the point) exists, (*) is summable $(\nu^2, k+\frac{1}{2}+\epsilon)$, $\epsilon>0$, to a sum equal to the derivative. *A. Zygmund* (Chicago, Ill.).

Stepanov, V. V. On the metric in the space of S_2 almost periodic functions. Doklady Akad. Nauk SSSR (N.S.) 64, 171–174 (1949). (Russian)

Stepanov, V. V. On a class of almost periodic functions. Doklady Akad. Nauk SSSR (N.S.) 64, 297–300 (1949). (Russian)

The author proves that the Stepanov metric of the functional space L^2, corresponding to the norm $\|f\|_\delta$ defined by $\|f\|_\delta^2 = \sup_\alpha \frac{1}{2}\delta^{-1}\int_{\alpha-\delta}^{\alpha+\delta}|f(t)|^2dt$, is equivalent to the more general metric which corresponds to the norm $\|f\|_{p(t)}$, defined by $\|f\|_{p(t)}^2 = \sup_\alpha \int_{-\infty}^{\infty}|f(t+\alpha)|^2p(t)dt$, where $p(t)$ is a positive measurable function satisfying the conditions: (1) the effective upper bound e.u.b.$p(t)$ is finite, (2) $\sum_{n=-\infty}^{\infty}$ e.u.b.$_{2n-1\leqq t\leqq 2n+1}\, p(t)$ is finite and (3) $\int_{-\infty}^{\infty}p(t)dt>0$. For an exponential sum $P_N(t) = \sum_1^N a_ne^{i\lambda_nt}$ there is obtained $\|P_N\|_{p(t)}^2 = \sup_\alpha \sum_{m,n=1}^N \varphi(\lambda_m-\lambda_n)a_me^{i\lambda_m\alpha}\bar{a}_ne^{-i\lambda_n\alpha}$, where

$$\varphi(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t}p(t)dt.$$

From this it follows easily that the infinite Hermitian form $\Phi(x_1, x_2, \cdots) = \sum_{m,n=1}^{\infty}\varphi(\lambda_m-\lambda_n)x_m\bar{x}_n$, where $\lambda_1, \lambda_2, \cdots$ are given real numbers, will be bounded for all functions $\varphi(\lambda) = \int_{-\infty}^{\infty}e^{i\lambda t}p(t)dt$, where $p(t)$ satisfies the above conditions, if it is bounded for one particular choice of the function $\varphi(\lambda)$.

In the second paper the author considers the special sequences $\lambda_1, \lambda_2, \cdots$, for which the Hermitian form $\Phi(x_1, x_2, \cdots)$ is bounded. It is proved that the Riesz-Fischer theorem holds for the class of almost periodic functions in the sense of Stepanov with exponents $\lambda_1, \lambda_2, \cdots$. A necessary and sufficient condition that $\Phi(x_1, x_2, \cdots)$ is bounded is that there exists a number K such that no interval of length 1 contains more than K numbers of the sequence $\lambda_1, \lambda_2, \cdots$.

H. Tornehave (St. Johns, Que.).

Valiron, Georges. Sur un théorème de Poincaré. C. R. Acad. Sci. Paris 228, 43–44 (1949).

The author gives a simple proof of the following theorem by Poincaré, who stated it without proof [Bull. Astr. 1, 319–327 (1884)]. A necessary (and sufficient) condition for the series $\sum C_n \sin \alpha_nt$ ($\alpha_n>0$, t real) to converge absolutely for all t is that the series $\sum'|C_n|$ and $\sum''|C_n\alpha_n|$, with summation over the n's for which respectively $\alpha_n>$ some fixed $\lambda>0$ and $\alpha_n\leqq\lambda$, are both convergent. The conclusion is obtained from the assumption of absolute convergence in a finite interval. The note contains also some remarks on the growth and zeros of the derivative $F'(t)$ of the entire function $F(t) = \sum''C_n \sin \alpha_nt$ (t now complex), when $\sum''|C_n| = \infty$.

E. Følner (Copenhagen).

Krein, M., and Levin, B. On entire almost periodic functions of exponential type. Doklady Akad. Nauk SSSR (N.S.) 64, 285–287 (1949). (Russian)

An almost periodic function $f(x)$ is said to be of exponential type if it has a largest Fourier exponent Λ and a smallest Fourier exponent $-\Lambda$. It will possess an analytic continuation $f(z) = f(x+iy)$ almost periodic in every strip $c<y<d$. Its zeros can be arranged in a sequence $\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots$ according to the size of their real

parts, and we have then $\Re a_n = (\pi/\Delta)n + \varphi(n)$, where $\varphi(x)$ is almost periodic. We have further $|\Im a_n| \leqq M$, but $\Im a_n$ is not always almost periodic. The function $f(z)$ can be written as a (principal value) product $f(z) = c\prod_{n=-\infty}^{\infty}(1-z/a_n)$ (where $1-z/0$ means z). The author announces the following theorems. If $a_n = (\pi/\Delta)n + \psi(n)$, where $\psi(x)$ is almost periodic with absolutely convergent Fourier series, then the product $\prod_{n=-\infty}^{\infty}(1-z/a_n)$ is an almost periodic function of exponential type with absolutely convergent Fourier series. If on the other hand $f(z)$ is an almost periodic function of exponential type and with absolutely convergent Fourier series and $|a_n - a_{n-1}| \geqq \alpha > 0$ for every n, then $a_n = (\Delta/\pi)n + \psi(n)$, where $\psi(x)$ is an almost periodic function with absolutely convergent Fourier series.

H. Tornehave (St. Johns, Que.).

Lee, Kwok-Ping. Sur les séries de Fourier et les classes quasi-analytiques des fonctions presque-périodiques. Quart. J. Sci. Wu-Han Univ. 9, 1–16 (1948).

A typical theorem is the following. The class of real almost periodic functions $f(t)$ with average value zero with a development of the form

$$\sum_{0<\lambda_n\to\infty} (a_n \cos \lambda_n t + b_n \sin \lambda_n t),$$

$\lim n/\lambda_n = 0$, $\liminf (\lambda_{n+1} - \lambda_n) = q > 0$, having a modulus of continuity $\omega(\delta)$ such that $\int_0 \delta^{-1}\omega(\delta)d\delta < \infty$, is quasi-analytic in the sense that if $f(t)$ is zero in an interval, then it vanishes identically. These theorems are obtained by combining theorems which by their conditions guarantee the analyticity of $\sum(a_n + ib_n)e^{-\lambda_n z}$ in an interval of $\Re z = 0$ in which $f(t) = 0$ with theorems which by their conditions guarantee the existence of singular points of this Dirichlet series in an interval of a certain length. *František Wolf.*

Kawata, Tatsuo. The characteristic function of a probability distribution. Tôhoku Math. J. 48, 245–256 (1941).

The characteristic function $f(t) = \int_{-\infty}^{\infty} e^{itx} d\sigma(x)$ is considered, where $\sigma(x)$ is a distribution function. When the integral above is taken from $-\alpha$ to α the resulting function is called $D_\alpha f$. It is shown that this is identical with the Dirichlet integral of $f(t)$ using the kernel $\{\sin \alpha(t-x)\}/\{\pi(t-x)\}$, where the Dirichlet integral is taken as a Cauchy principal value over $(-\infty, \infty)$. In case the spectrum is bounded, i.e., $\sigma(x)$ is constant outside of $(-\alpha, \alpha)$, then clearly $f(x)$ is an entire function of exponential type. The author uses a theorem of Wiener and Paley to show that $f(x)$ cannot be too small for large $|x|$. [A somewhat stronger result than the author gives is true. Indeed by a well-known theorem of Carleman it follows that if the spectrum is bounded on one side then $\int_{-\infty}^{\infty} |\log |f(x)||/(1+x^2)dx < \infty$.] The integral, over $(1, T)$ as $T\to\infty$, of $f(x)e^{it x}/x$ is shown to exist, as is the one over $(-T, -1)$, when $(\sigma(\xi+x) - \sigma(\xi-x))/x$ is integrable over $0<x<c$. Finally a theorem of Khintchine on unimodal distribution functions is considered.

N. Levinson (Cambridge, Mass.).

Orlicz, W. Sur la convergence uniforme des développements orthogonaux de fonctions bornées. Colloquium Math. 1, 218–224 (1948).

The space (R) is the metric space of functions $x(t)$ measurable in $[a, b]$ and with $|x(t)| \leqq 1$ essentially, the distance between two elements $x_1(t)$ and $x_2(t)$ being defined by $\int_a^b |x_1(t) - x_2(t)| dt$. Let the $\varphi_i(t)$ form a normal orthogonal system of essentially bounded measurable functions over $[a, b]$; and let T be a regular Toeplitz method of summation corresponding to a row-finite matrix. It is proved that the set of all functions $x(t)$ of (R), whose Fourier series with respect to the $\varphi_i(t)$ are essentially uniformly summable by T, is of the first category in (R). The proof is based on general theorems on linear operations in (R).

W. W. Rogosinski (Newcastle-upon-Tyne).

Pollard, Harry. The mean convergence of orthogonal series. III. Duke Math. J. 16, 189–191 (1949).

[For parts I and II see Trans. Amer. Math. Soc. 62, 387–403 (1947); 63, 355–367 (1948); these Rev. 9, 280, 426.] Given numbers $\alpha \geqq -\frac{1}{2}$, $\beta \geqq -\frac{1}{2}$, let

$$M(\alpha, \beta) = 4 \max \left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\},$$

$$m(\alpha, \beta) = 4 \min \left\{\frac{\alpha+1}{2\alpha+1}, \frac{\beta+1}{2\beta+1}\right\}$$

and let $f(x) \sim \sum a_n p_n(x)$ be the development of $f(x)$, $-1 \leqq x \leqq 1$, into the series of Jacobi polynomials corresponding to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$. It is shown that, if $M(\alpha, \beta) < p < m(\alpha, \beta)$ and if $|f|^p w(x) \epsilon L$, then

$$\lim_{N\to\infty} \int_{-1}^{1} |f(x) - \sum_0^N a_n p_n(x)|^p w(x) dx = 0.$$

The conclusion is false for p outside the interval (m, M). The cases of $p = M$ and $p = m$ remain open.

A. Zygmund (Chicago, Ill.).

Kozlov, V. Ya. On a local characteristic of complete normal orthogonal systems of functions. Mat. Sbornik N.S. 23(65), 441–474 (1948). (Russian)

If $\Delta = [x_1, x_2]$ is an interval, let $\Delta F = F(x_2) - F(x_1)$ and let $\Delta(x)$ be the characteristic function of Δ. We say that $\{F_n(x)\} \epsilon P(a, b)$, if for any two intervals $\Delta_1, \Delta_2 \subset [a, b]$ we have $\sum_{n=1}^{\infty} \Delta_1 F_n \Delta_2 F_n = \text{mes}\,(\Delta_1 \cap \Delta_2)$. If for suitable definition of the $F_n(x)$ outside $[a, b]$ we have $\{F_n\} \epsilon P(a', b')$ in a larger interval $[a', b']$ containing $[a, b]$, then we say that $\{F_n\}$ can be continued to $[a', b']$. The following properties of $\{F_n\} \epsilon P$ are established. (i) (1) $F_n(x) = \int_a^x l_n(x)dx + c_n$, where $l_n(x) \epsilon L^2(a, b)$. (ii) If $A = (a_{ij})$ is an orthogonal matrix and $\{\varphi_n\}$ is an orthonormal, complete set in $L^2(a, b)$, then for every $q > 0$

$$\left\{F_n(x) = \sum_{k=1}^{\infty} a_{k+q,n} \int_a^x \varphi_k(x)dx\right\} \epsilon P[a, b] \tag{2}$$

and cannot be continued to a larger interval. The set $\{F_n'(x)\}$ is orthonormal, if and only if $q = 0$. Vice versa, if $\{F_n\} \epsilon P(a, b)$ and cannot be continued to a larger interval, then $\{F_n\}$ is of the form (1). (iii) If $\{F_n\}$ can be continued to $[a_2, b_2] \supset [a, b]$, then the continuation can be carried out in such a way that $\{F_n'(x)\}$ is an orthogonal set in $[a_1, b_1]$, $[a_2, b_2] \supset [a_1, b_1] \supset [a, b]$.

The proofs are based on the consideration of the transformation $T(\Delta(x)) = \sum \Delta F_n \varphi_n(y)$, where $\{\varphi_n\}$ is a complete, orthonormal system in $L^2(0, 1)$. Then T maps the characteristic functions $\Delta(x)$ of intervals isometrically in $L^2(a, b)$. The transformation can be extended to an isometric transformation T of $L^2(a, b)$ in $L^2(0, 1)$. If $T(f) = \sum c_n(f)\varphi_n(y)$, then $c_n(f)$ is a linear functional in $L^2(a, b)$ and therefore $c_n(f) = \int_a^b f(x) l_n(x) dx l_n(x) \epsilon L^2(a, b)$. Putting $f(x) = \Delta(x)$ proves (1). Vice versa, if $T(f) = \sum \varphi_n(y) \int_a^b f(x) l_n(x) dx$ is an isometric mapping of $L^2(a, b)$ in $L^2(0, 1)$, then (1) defines a set $\{F_n\} \epsilon P(a, b)$.

The choice $l_n=\sum_k a_{k+q,n}\varphi_k$ leads to (2). The converse result is established by proving first that $\{F_n\}$ can be continued, if the subspace S orthogonal to all $T(f)$ is of infinite dimension. If S has a finite dimension q, then we can find a complete orthogonal system $\{h_n(y)\}$ in $L^2(0, 1)$ such that $h_1, \cdots, h_q$ span S, and such that $h_k=T(\psi_k)$ for $k>q$, where $\{\psi_k\}$ is a complete orthonormal set in $L^2(a, b)$. Then

$$T(f)=\sum_{q+1}^{\infty}h_k(y)\int_a^b f(x)\psi_k(x)dx$$

and the result follows on expressing h_k in terms of the φ_n.

Carleman introduced measurable functions $K(x, y)$ $(0\leqq x\leqq 1,\ 0<y<1)$ with the following properties: (a) $K(x, y)\varepsilon L^2(0, 1)$ for every fixed x,

$$\text{(b)}\quad \int_0^1 \{K(x_2, y)-K(x_1, y)\}\{K(x_2', y)-K(x_1', y)\}dy = \operatorname{mes}(\Delta\cap\Delta'),$$

where $\Delta=(x_1, x_2)$, $\Delta'=(x_1', x_2')$. As an application of his theorems the author shows that a function $K(x, y)$ satisfies these conditions, if and only if there are two orthonormal sets $\{\theta_n(x)\}$, $\{\varphi_n(x)\}$ such that $\{\theta_n(x)\}$ is complete and

$$K(x, y)=\sum_{n=1}^{\infty}\int_0^x \theta_n(x)dx\varphi_n(y)$$

almost everywhere. *W. H. J. Fuchs* (Ithaca, N. Y.).

Kozlov, V. Ya. On the distribution of positive and negative values of normal orthogonal functions forming a complete system. Mat. Sbornik N.S. 23(65), 475–480 (1948). (Russian)

Theorem. If $\{\varphi_n(x)\}$ is an orthonormal system complete in $L^2(0, 1)$, then $\sum_n(\varphi_n^+(x))^2$ and $\sum(\varphi_n^-(x))^2$ diverge almost everywhere. Here $\varphi^+=\max(\varphi, 0)$, $\varphi^-=\min(\varphi, 0)$. The proof is based on the following lemma. Let P be a perfect set such that every interval in $(0, 1)$ has a set of positive measure in common with P, and $\{s_n(x)\}$ a sequence of continuous functions converging on P in the mean to a bounded function $f(x)$. If for an $\epsilon>0$ there is a set $Q\subset P$ of the second category such that every η-neighborhood of an $x\varepsilon Q$ contains two sets $P_1, P_2\subset P$ of positive measure with $\inf_{x\varepsilon P_1} f(x)-\sup_{x\varepsilon P_2} f(x)>\epsilon$, then the sequence $\{s_n(x)\}$ diverges in a set $R\subset P$ of the second category. This lemma is also used to prove that, to any orthonormal system complete in $L^2(0, 1)$, one can construct a function $g(x)\varepsilon L^2(0, 1)$ whose Fourier series with respect to this system diverges in a set of the second category. *W. H. J. Fuchs.*

Povzner, A. On the completeness of the sequence of functions $e^{\Lambda_n t}$ in $L^2(-\pi, \pi)$. Doklady Akad. Nauk SSSR (N.S.) 64, 163–166 (1949). (Russian)

Let $\{\lambda_n\}$ $(-\infty<n<\infty)$ be a sequence of complex numbers, $\alpha_p=\Re(\lambda_p-p)$, $\beta_p=\Re(\lambda_{-p}+p)$ $(p>0)$. Then $\{e^{\Lambda_n t}\}$ is complete in $L^2(-\pi, \pi)$, if (1) $|\lambda_n-n|<A$ $(-\infty<n<\infty)$, (2) $\limsup \alpha_n<\frac{1}{2}$, $\limsup \beta_n<\frac{1}{2}$ and

$$L=\limsup n^{-1}\sum_{p=1}^{n}(\alpha_p-\beta_p)<\tfrac{1}{2}.$$

This is an improvement of results of Levinson [Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., v. 26, New York, 1940, p. 6; these Rev. 2, 180]. The proof is based on choosing constants p_k such that

$$e^{\Lambda t}-\sum_{k=-n}^{n}p_k e^{\Lambda_k t}=\sum_{|k|>n}q_{k,n}(\lambda)e^{ikt},\qquad |t|<\pi,$$

and then proving by elementary calculation that under the hypothesis of the theorem

$$|q_{k,n}(\lambda)|=O(n^L(|k|+n)/|k|(|k|-n)),$$

so that $\|e^{\Lambda t}-\sum_{k=-n}^{n}p_k e^{\Lambda_k t}\|\rightarrow 0$ as $n\rightarrow\infty$. [For a related method see O. Szász, Math. Ann. 77, 482–496 (1916).]

W. H. J. Fuchs (Ithaca, N. Y.).

Good, I. J., and Reuter, G. E. H. Bounded integral transforms. Quart. J. Math., Oxford Ser. 19, 224–234 (1948).

The authors present a unified treatment of Watson's theory of general transforms in $L^2(0, \infty)$ and the subsequent generalizations. The main tool is the "standard" kernel $\phi(x, t)$, defined on $0\leqq x<\infty$, $0\leqq t<\infty$ by the requirements that $\phi(0, t)=0$ for almost all t, and that there exists a constant B such that for any k_n, a_n, b_n $(n=1, \cdots, N;$ $0\leqq a_1\leqq b_1\leqq\cdots\leqq a_N\leqq b_N)$ the inequality

$$\int_0^\infty\left\{\sum_1^N k_n\{\phi(b_n, t)-\phi(a_n, t)\}\right\}^2 dt\leqq B^2\sum_1^N k_n^2(b_n-a_n)$$

holds. The unification is based on the fact that the linear transformation of L^2 defined by $(d/dx)\int_0^\infty f(t)\varphi(x, t)dt$ is bounded if and only if ϕ is standard. The treatment avoids Hilbert space theory almost completely. The authors call particular attention to the work of G. Temple [Proc. London Math. Soc. (2) 31, 243–252 (1930)] which has been overlooked by subsequent writers. *H. Pollard.*

Polynomials, Polynomial Approximations

Marden, Morris. The zeros of certain real rational and meromorphic functions. Duke Math. J. 16, 91–97 (1949).

Let $f(z)$ and $f_1(z)$ be real polynomials whose ratio $F_1(z)=f_1(z)/f(z)$ has a partial fraction development $F_1(z)=\sum_{j=1}^{n}\gamma_j/(z-c_j)$, $c_j=a+ib_j$, $\gamma_j=\alpha_j+i\beta_j=m_je^{i\varphi_j}$, $\alpha_j\neq 0$. For the zeros of $f_1(z)$ generalizations of Rolle's theorem and Jensen's theorem and some of Walsh's theorems are obtained. This is done by the replacement of the Jensen circles of $f(z)$ by the circles $K(c_j, \mu_j)$ of $F_1(x)$, each of which passes through the conjugate imaginary zeros c_j and c_j^*, and has center $z=k_j$ (real) such that the angle c_j^*, c_j, k_j is μ_j. These results are extended to systems of rational functions $F_1(z)$ and to meromorphic functions of similar form.

E. Frank (Chicago, Ill.).

Sz. Nagy, Gyula. Über den Wertvorrat gebrochener rationaler Funktionen in Kreisbereichen. Hungarica Acta Math. 1, no. 3, 1–13 (1948).

Let $n_0=0$, $n_1=1$ and $1<n_2<\cdots<n_k$; and let $f(z)=\sum_0^p a_k z^{n_k}$, $g(z)=\sum_0^p b_k z^{n_k}$ and $F(z)=f(z)/g(z)$. Let C be a circular region which contains the origin and let K be the region upon which C is mapped by $Z=F(z)$. By use of Laguerre's theorem the author shows that, if $D=a_0b_1-a_1b_0\neq 0$, the region K always contains the circular regions C' and C'' upon which C is mapped by $Z=(pa_0+a_1z)/(pb_0+b_1z)$ and $Z=(qa_0+a_1z)/(qb_0+b_1z)$ with $q=\prod_2^p n_k/(n_k-1)$, respectively; that is, if Z is any point of C' or C'', then $F(z)=Z$ for at least one z in C. If $D=0$ but $a_0b_0a_1b_1\neq 0$, $F(z)$ assumes every value in C provided C also contains at least one of the points $-na_0/a_1$, $-pa_0/a_1$ and $-qa_0/a_1$. The author calls attention to the fact that, if at least $m\leqq n_p$ zeros of $f(z)$ are known to lie in a circle $|z|\leqq R=R(a_0, a_1, \cdots, a_m)$ for all choices of the a_k, $k>m$, and if $b_k=\lambda a_k$ for $k=0, 1, \cdots, m$,

then $F(z)$ assumes every value at least p times in the circle $|z|\leqq R$. By using some well-known choices of R, he is then able to specify various circles within which the function $F(z)$ assumes every value at least m times.

M. Marden (Milwaukee, Wis.).

Tomić, Miodrag. Généralisation et démonstration géométrique de certains théorèmes de Fejér et Kakeya. Acad. Serbe Sci. Publ. Inst. Math. **2**, 146–156 (1948). (French. Serbian summary)

The Eneström-Kakeya theorem that the real polynomial $f_n(z)=\sum_0^n a_j z^j\neq 0$ for $|z|<1$ if $a_0\geqq a_1\geqq\cdots\geqq a_n>0$ is proved in this paper by construction of the polygonal line joining in succession the points $Z_0=0$, $Z_k=f_k(re^{i\theta})$, with r fixed, $0<r<1$. The same method is used to prove a theorem attributed to Karamata, covering the possibility of zeros on $|z|=1$. From the Kakeya theorem, the author deduces the theorem that, if $c_{m-j+1}-c_{m+j}\geqq c_{m-j}-c_{m+1+j}>0$ for $j=1, 2, \cdots, m=n/2$ or $(n-1)/2$, then $f_n(z)\neq 0$ for $|z|=1$, $z\neq 1$. The latter theorem is shown to specialize to two theorems on trigonometric polynomials due to Fejér.

Reviewer's note. The second theorem attributed to Karamata was proved in 1913 by A. Hurwitz [Tôhoku Math. J. **4**, 89–93 (1913) = Werke, v. 2, Basel, 1933, pp. 626–631]. Also it is incorrectly stated, the correct version being: $f_n(z)$ has $k-1$ zeros on $|z|=1$ if and only if $n+1$ has a factor k such that $a_j=a_{j-1}$ whenever k is not a factor of j, $j=1, 2, \cdots, n$. The third theorem is also not new; it is a special case of a result derived by Egerváry [Acta Litt. Sci. Szeged **5**, 78–82 (1931)]. *M. Marden* (Milwaukee, Wis.).

Karamata, J., et Tomić, M. Considérations géométriques relatives aux polynômes et séries trigonométriques. Acad. Serbe Sci. Publ. Inst. Math. **2**, 157–175 (1948). (French. Serbian summary)

The results and geometric methods of the paper reviewed above are applied to trigonometric polynomials and series. The following are typical results. (I) If the a_k $(k=0, 1, \cdots, n)$ form a nonincreasing sequence, then for all real p and q and $0<x<\pi$ the polynomial $S(x)=\sum_0^n a_k \sin(kp+q)x$ satisfies the inequality $-a_0\sin^2 X\csc(px/2)\leqq S_1(x)\leqq a_0\cos^2 X\csc(px/2)$, where $4X=(2q-p)x$. (II) If the a_k $(k=0, 1, \cdots, n)$ form a convex sequence (i.e., if $a_{k+1}-2a_k+a_{k-1}\geqq 0$ for $k=1, 2, \cdots, n$, with $a_{n+1}=a_{-1}=0$), then the polynomial

$$C(x)=\tfrac{1}{2}a_0+\sum_1^n a_k\cos kx$$

satisfies the inequality $0\leqq 2C(x)\leqq(a_0-a_1)\csc^2(x/2)$ for $0<x<2\pi$. (III) If the a_k $(k=1, 2, \cdots)$ form a nonincreasing sequence with $a_k\to 0$ as $k\to\infty$, then there are always two positive numbers c and h such that the series $s(x)=\sum_1^\infty a_k\sin kx$ satisfies the inequality $s(x)\geqq cx$ for all x, $0<x<h$. *M. Marden* (Milwaukee, Wis.).

Markovitch, D. Sur la limite inférieure des modules des zéros d'un polynôme. Acad. Serbe Sci. Publ. Inst. Math. **2**, 236–242 (1948). (French. Serbian summary)

Let $g(r)=\sum_1^\infty b_k r^k$, with $b_k>0$ for all k, be convergent for $|r|<R$, R sufficiently large. For any zero $z=re^{i\theta}$ of a given polynomial $f(z)=\sum_0^n a_k z^k$, one has the inequality $|a_0|\leqq\sum_1^n|a_k|r^k$ which, if divided by the inequality $g(r)\geqq\sum_1^n b_k r^k=g_n(r)$ yields the inequality

$$|a_0|g(r)^{-1}\leqq G=\left\{\sum_1^n|a_k/b_k|b_k r^k\right\}/g_n(r)\leqq M,$$

where $M=\max|a_k/b_k|$ for $k=1, \cdots, n$; for G is a mean of the quantities $|a_k/b_k|$ for $k=1, \cdots, n$. If now the b_k are so chosen that the equation $g(r)=|a_0|/M$ has a single positive root r_0, this root will be a lower bound for moduli of the zeros of $f(z)$. The author illustrates this principle with some examples such as $b_k=t^{-k}$, $t>0$, which leads to the result of Landau [Tôhoku Math. J. **5**, 97–116 (1914)] that $r_0=|a_0|t/(|a_0|+M)$, where $M=\max(|a_k|t^k)$ for $k=1, \cdots, n$. *M. Marden* (Milwaukee, Wis.).

Demontvignier, Marcel, et Lefèvre, Paul. Généralisation du critérium de stabilité de Nyquist. C. R. Acad. Sci. Paris **228**, 360–362 (1949).

The following generalization of the Nyquist criterion is derived. Let the characteristic equation of a linear system be written in the form $R(p)-A=0$, where A is a real parameter and where $R(p)=F_r(p)/F_s(p)$ with $F_r(p)$ and $F_s(p)$ relatively prime real polynomials of degrees r and s, respectively. Let P be the number of poles of $R(p)$ in the half-plane $\Re(p)>0$, m the order of the origin as a pole and n the sum of the orders of the poles on the positive imaginary axis. Let $4N=-m-2n-(r-s)-2P$ when $r>s$ and $4N=-m-2n-2P$ when $r\leqq s$. Then for the system to be stable it is necessary and sufficient for the point $w=R(iy)$ to wind N times (net) at a finite distance around the point $w=A$ as y varies from 0 to ∞. [Reviewer's note. Both the Nyquist criterion and the above generalization are essentially included in Cauchy's principle of argument.]

M. Marden (Milwaukee, Wis.).

Demontvignier, Marcel, et Lefèvre, Paul. Étude de la stabilité d'un système linéaire à partir du diagramme de phase généralisé. C. R. Acad. Sci. Paris **228**, 463–465 (1949).

[Cf. the note reviewed above.] The generalized phase diagram is the graph of $\arg R(p)$ as p varies on a contour consisting of the positive imaginary axis, a quadrant of a large circle about the origin and the positive real axis. This graph has discontinuities at the poles of $R(p)$ on the positive imaginary axis. The behavior of the graph and of $|R(p)|$ between successive discontinuities permits one to determine the value of $Z-P$ where Z is the number of zeros of $R(p)$ in the half-plane $\Re(p)>0$. *M. Marden.*

Cotton, E., et Yuan, Ma Min. Sur les critères de stabilité de Routh et de Hurwitz. Bull. Sci. Math. (2) **72**, 115–128 (1948).

In extension of the criteria of Routh and Hurwitz for all the zeros of a real polynomial to have negative real parts, the author develops some theorems for determining the number of zeros of a complex polynomial $f(z)=\sum_0^n(a_k+ib_k)z^{n-k}$ in the half-plane $\Re(z)<0$. The theorems are derived by the use of the Cauchy indices, Sturm sequences and Sylvester's elimination method. [Reviewer's note. Equivalent results were derived differently by Bilharz, Z. Angew. Math. Mech. **24**, 77–82 (1944); these Rev. **7**, 62 and by Frank, Bull. Amer. Math. Soc. **52**, 144–157 (1946); these Rev. **7**, 295.]

M. Marden (Milwaukee, Wis.).

Berman, D. L. The convergence of certain interpolation operations. Doklady Akad. Nauk SSSR (N.S.) **64**, 5–8 (1949). (Russian)

Let us consider functions $f(x)$ defined for $-1\leqq x\leqq 1$ and their Lagrange interpolating polynomials associated with fundamental points $-1\leqq x_1^n<x_2^n<\cdots<x_n^n\leqq 1$ $(n=1, 2, \cdots)$.

While it is classical that in the case of the Chebyshev abscissas ($x_k^n = \cos(2k-1)\pi/2n$) the interpolating polynomials may diverge even if f is continuous, S. Bernstein showed [Comm. Soc. Math. Kharkow (4) 5, 49–57 (1932)] that we obtain uniform convergence of these polynomials to any continuous $f(x)$ if in the polynomials we replace the values $f(x_k^n)$ by (*) $\frac{1}{4}\{f(x_{k-1}^n)+2f(x_k^n)+f(x_{k+1}^n)\}$ for $k=2, 3, \cdots, n-1$ and by $\frac{1}{4}\{3f(x_1^n)+f(x_2^n)\}$, $\frac{1}{4}\{f(x_{n-1}^n)+3f(x_n^n)\}$ for $k=1$ and $k=n$, respectively. This result is extended in the present paper to more general systems of points and to linear combinations of the $f(x_k^n)$ more general than (*), but the new results become too complicated to be stated here. The author also applies his generalizations to bounded measurable functions with $f(x_k^n)$ replaced by the averages

$$\int_{x_k^n}^{x_{k+1}^n} f(t)(x_{k+1}^n - x_k^n)^{-1}dt.$$

A. Zygmund (Chicago, Ill.).

Krall, H. L., and Frink, Orrin. A new class of orthogonal polynomials: The Bessel polynomials. Trans. Amer. Math. Soc. 65, 100–115 (1949).

The Bessel polynomials of degree n are introduced by the formula

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!}(x/2)^k.$$

They satisfy the differential equation

$$x^2y''+(2x+2)y' = n(n+1)y,$$

and are related to the wave equation in space. An integral property is proved which can be interpreted as orthogonality and also as an analogue of Rodrigues' formula. Recurrence formulas and a generating function are also established. The Bessel polynomials can be expressed in terms of the Bessel functions of order $n+\frac{1}{2}$. A generalization of the Bessel polynomials is also studied. *G. Szegö.*

Special Functions

★Schelkunoff, S. A. Applied Mathematics for Engineers and Scientists. D. Van Nostrand Company, Inc., New York, N. Y., 1948. xi+472 pp. $6.50.

Most of this book is devoted to topics in advanced calculus. In addition, there are chapters on the gamma function, exponential integrals, Fresnel integrals, Bessel functions and Legendre functions which form a convenient compilation of formulas, and some short numerical tables, connected with these special functions.

R. P. Boas, Jr. (Providence, R. I.).

Wise, M. E. The incomplete beta function and the incomplete gamma function: An acknowledgment. J. Roy. Statist. Soc. Ser. B. 10, 264 (1948).

The author acknowledges that a result of his [Suppl. J. Roy. Statist. Soc. 8, 202–211 (1946), in particular, pp. 209–210; these Rev. 9, 49] is contained in earlier work by E. C. Molina [Bell System Tech. J. 11, 563–575 (1932)].

Kourganoff, Vladimir. Sur les transformées, par les opérateurs Λ et Φ, des fonctions intégro-exponentielles K_n. C. R. Acad. Sci. Paris 227, 958–960 (1948).

The author obtains expansions for

$$2\lambda_n = \int_0^\infty K_n(t)K_1(|t-\tau|)dt$$

and

$$\tfrac{1}{2}\varphi_n = \int_0^\infty \operatorname{sgn}(t-\tau)K_n(t)K_2(|t-\tau|)dt,$$

where K_n are the functions which he has already studied and tabulated [same C. R. 225, 430–431 (1947); Ann. Astrophysique 10, 282–299, 329–340 (1947); these Rev. 9, 91, 349, 432]. He also gives short numerical tables for λ_n and φ_n for $n=2, 3, 4$, and of two auxiliary functions. These results are required in work on Milne's integral equation of the theory of radiative equilibrium and neutron diffusion theory. *A. Erdélyi* (Pasadena, Calif.).

Meixner, Josef. Entwicklung eines Produktes zweier Kugelfunktionen mit verschiedenen Argumenten nach Kugelfunktionen. Arch. Math. 1, 173–181 (1948).

Comparing normal solutions of Laplace's equation in spheroidal coordinates with the solutions in spherical polars (the polar axis coincides with the axis of revolution of the spheroid, but the centre of the concentric spheres does not coincide with the centre of the spheroid), the author obtains the expansion

$$\begin{aligned} 2^{-\nu}e^{-\mu\pi i}\Gamma(\tfrac{1}{2}-\nu)\mathfrak{Q}_{-\nu-1}^{\mu}(\xi)\mathfrak{Q}_{\nu}^{\mu}(\eta) &= \Gamma(\tfrac{1}{2})\Gamma(\mu-\nu)\sum_{t=0}^{\infty}\frac{(-a)^t\xi^{\nu-t}}{t!\Gamma(\nu+\mu-t+1)} \\ &\quad \times {}_2F_1(-\tfrac{1}{2}t, -\tfrac{1}{2}t+\tfrac{1}{2}; \tfrac{1}{2}-\nu; a^{-2}) \\ &\quad \times \mathfrak{Q}_{\nu-t}^{\mu}\left(\frac{\xi\eta+a}{\zeta}\right), \end{aligned}$$

where $\zeta^2 = \xi^2+\eta^2+a^2+2\xi\eta a-1$, and $\mathfrak{Q}$ is the Legendre function of the second kind (defined for the cut complex plane). He investigates the convergence of the series and notes a number of special cases of his expansion.

A. Erdélyi (Pasadena, Calif.).

Buchholz, Herbert. Integral- und Reihendarstellungen für die verschiedenen Wellentypen der mathematischen Physik in den Koordinaten des Rotationsparaboloids. Z. Physik 124, 196–218 (1948).

Typical solutions of the wave equation in paraboloidal coordinates can be written down as products of confluent hypergeometric functions. The author aims at expressing all the most important wave forms as superpositions (infinite series or integrals) of paraboloidal waves. He starts with the most general cylindrical wave function which he represents both as an infinite series of, and as an infinite integral over, typical solutions in paraboloidal coordinates. The series representation is essentially the so-called Hille-Hardy theorem, the integral its continuous analogue [cf. the reviewer's paper, Proc. Roy. Soc. Edinburgh. Sect. A. 61, 61–70 (1941); these Rev. 3, 116]. Plane waves and spherical waves of the most general description have known expressions in terms of cylindrical waves: hence their expression in terms of paraboloidal waves. In the discussion of spherical waves it is assumed that the source of the waves is at the focus of the paraboloid, and that the polar axis coincides with the axis of revolution of the paraboloid. However, for the simplest type of (spherically symmetric) spherical wave the representation is obtained for the eccentric case too, when the wave centre need not coincide with the focus of the paraboloid. *A. Erdélyi* (Pasadena, Calif.).

Abramowitz, Milton. Asymptotic expansions of Coulomb wave functions. Quart. Appl. Math. 7, 75–84 (1949).

The solutions of $w''+(1-2\eta z^{-1})w=0$ are expanded asymptotically in reciprocal powers of z for z large and η moderate. For physical applications the author studies especially the solution F which for large z is approximately equal to $\sin(z-\eta\log 2z+\arg\Gamma(1+i\eta))$. He determines a number of zeros of this function. He gives an asymptotic expansion for $\log F$ (where η is large and $t=\frac{1}{2}z/\eta$ moderate) of the form $2\eta g_0(t)+g_1(t)+(2\eta)^{-1}g_2(t)+\cdots$, for $\log F$ (where $r=(2\eta z)^{\frac{1}{2}}$ is large and z moderate) of the form $2r-r^{-1}h_1(z)+r^{-3}h_2(z)+\cdots$ (the coefficients of which are polynomials in z) and further for F (where r is large and z moderate) of the form $\lambda(1+v_1r^{-1}+v_2r^{-2}+\cdots)$, where $\eta\lambda$ is an elementary function of r and v_h denotes a polynomial in z^3 of degree h. [Read in formula (6.3) $2\rho^{-\frac{1}{2}}u'_{n+1}+u''_n$ instead of $2\rho^{-\frac{1}{2}}u'_n+u''_{n+1}$.] If $F=\mu\sin H$, where μ denotes a suitable elementary function of η and $q=\frac{1}{2}z\eta^{-1}-1$, and if η is large and $q>0$, then he deduces for H an asymptotic expansion of the form $\frac{1}{4}\pi+2\eta h_0(q)+(2\eta)^{-1}h_1(q)+\cdots$. Moreover he expands F for small values of η in ascending powers of 2η and finally for large η in reciprocal powers of 2η, where the coefficients denote Bessel functions of r or related functions.

J. G. van der Corput (Amsterdam).

Emde, Fritz. Pfeil-Diagramme für Zylinderfunktionen. Z. Angew. Math. Mech. 28, 360–368 (1948). (German. Russian summary)

The author describes the Argand diagrams corresponding to some of the basic formulae of Bessel function theory and points out the practical usefulness of such diagrams. For real variable x and positive p, he shows how to construct in the complex plane $J_{-p}(x)$, $Y_{-p}(x)$, $H^{(1,2)}_{-p}(x)$ if $J_p(x)$ and $Y_p(x)$ are given. For an imaginary variable, a similar construction is given except that in this case one starts with given values of J_p and $H_p^{(1)}$. For an imaginary variable the author also shows how to construct diagrams describing the variation of J_p and $H_p^{(1)}$ when the variable is fixed and p varies. For unrestricted complex variable there are again the construction of all significant solutions from two linearly independent ones, and constructions for functions of variable $ze^{m\pi i}$, in particular for functions of a negative real variable.

A. Erdélyi (Pasadena, Calif.).

Bailey, W. N. Some integrals involving Hermite polynomials. J. London Math. Soc. 23, 291–297 (1948).

The integral $\int_{-\infty}^{\infty}e^{-x^2}H_m(ax)H_n(bx)dx$ can be evaluated in terms of an ordinary hypergeometric series. It can be evaluated in terms of gamma functions when $b=1$ or when $a^2+b^2=1$. The integral

$$\int_{-\infty}^{\infty}e^{-x^2}H_m(ax)H_n(bx)H_p(cx)dx$$

can be evaluated in terms of Lauricella's hypergeometric series of three variables. It can be evaluated as a finite double sum when $b=c$, $c=1$, $a^2+b^2=1$, or $a^2+b^2+c^2=1$; in terms of a ${}_3F_2$ when $a=b$ and $c=1$; in terms of an ordinary hypergeometric series when $b=c=1$, $a^2+b^2=1$ and $b=c$, or $a^2+b^2=1$ and $c=1$; and in terms of gamma functions when $a=b=c=1$ or $a=b=c=2^{-\frac{1}{2}}$. A few of these cases were known, but the majority are thought to be new.

A. Erdélyi (Pasadena, Calif.).

von Schelling, Hermann. A formula for the partial sums of some hypergeometric series. Ann. Math. Statistics 20, 120–122 (1949).

The complementary probabilities of having and not having the n_1th black ball appear at the latest in the nth drawing from an urn following a Pólya-Eggenburger scheme are used to derive an identity for corresponding partial sums of hypergeometric series, whose interest is chiefly that one sum has fewer terms than the other and is easier to compute.

J. Riordan (New York, N. Y.).

Burchnall, J. L. On the well-poised ${}_3F_2$. J. London Math. Soc. 23, 253–257 (1948).

By means of transformations of the corresponding differential equation, the well-poised hypergeometric function ${}_3F_2$ of argument x is expressed in terms of similar well-poised ${}_3F_2$ of argument $-(x-1)^2/4x$. Thus it is shown that the well-poised hypergeometric differential equation of order 3 is completely solvable in terms of functions ${}_3F_2$ in the neighborhood of all its singularities.

N. A. Hall.

Inui, Teturo. Unified theory of recurrence formulas. II. Progress Theoret. Physics 3, 244–261 (1948).

The first part appeared in the same vol., 168–187 (1948) [these Rev. 10, 296]. In the present second part the confluent hypergeometric function is similarly discussed and eight "stair operators" of the form $z^ae^{cz}(d/dz)z^be^{kz}$ are used.

A. Erdélyi (Pasadena, Calif.).

Differential Equations

Bruwier, L. Sur l'intégration des systèmes d'équations différentielles linéaires, à coefficients constants. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 3, 532–542 (1948).

The author points out formulations involving series or complex integrals as an alternative to the operational solution of equations with constant coefficients.

P. Franklin (Cambridge, Mass.).

Péyovitch, T. Sur l'intégration d'un système d'équations différentielles. Acad. Serbe Sci. Publ. Inst. Math. 2, 176–189 (1948). (French. Serbian summary)

For a system $dx_i/dt=\sum_k a_{ik}(t)e^{x_k}$, some very special cases are explicitly integrated.

P. Franklin.

Aubert, Marius. Sur une solution de l'équation de Fourier. C. R. Acad. Sci. Paris 228, 816–817 (1949).

Solutions of (*) $y''+2xy'+2\alpha y=0$ are written down in terms of integrals. In case the constant α is a positive or negative integer an association between a solution of (*) and a Hermite polynomial is pointed out.

F. G. Dressel (Durham, N. C.).

Mitrinovitch, D. S. Transformation d'une équation différentielle. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire 1, 97–114 (1948). (Serbian. French and Russian summaries)

Let $p\neq 0$, $p+q\neq 0$ be integers and $a_1\neq b_1$, a_2, b_2, $f\neq 0$ be functions of x. A rational transformation on the dependent variable y is written down which transforms

$$(*)\qquad (dy/dx+a_1y+a_2)^p(dy/dx+b_1y+b_2)^q=f(x)$$

into an equation of the form

$$(**)\qquad \frac{d\lambda}{dx}=\frac{\alpha_0\lambda^{p+q+1}+\alpha_1\lambda^{p+1}+\alpha_2\lambda}{\beta_0\lambda^{p+q}+\beta_1}.$$

The author also gives necessary conditions so that equations of type (**) can be transformed to those of type (*).
F. G. Dressel (Durham, N. C.).

Gremyačenskiĭ, A. P. Generalization of a theorem of Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 667–668 (1948). (Russian)

A theorem of Liapounoff states that the system of differential equations $dx/dt=P(t)x$, where x is a vector and $P(t)$ is a periodic matrix, is solvable by n quadratures if there exists a matrix B with constant coefficients such that $BP(t)+P(-t)B=0$ and such that the characteristic roots of B are all distinct as are their squares. The author shows that the matrix B may have repeated roots providing there is at most one independent vector solution v to the equation $(B-\lambda)v=0$ for any fixed λ. *N. Levinson.*

Manacorda, T. Vibrazioni forzate di un particolare sistema oscillante non lineare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 557–561 (1948).

Consideration of a simple mechanical system leads to the differential equation $d^2x/d\tau^2+\Gamma^2x=F(\tau)+B^2x(1+x^2)^{-\frac12}$, where Γ and B are constants, and $F(\tau)$ is an odd periodic function with the period 2π. The author obtains the following two results concerning this equation. (1) If $0<\Gamma\leqq\frac12$ and $0<B^2<\Gamma^2$, the equation has a unique periodic solution with the period 2π. (2) If Γ is positive, and not equal to an integer, there exists a positive number B_0 such that if $-B_0\leqq B\leqq B_0$, the equation has a unique periodic solution with the period 2π. *L. A. MacColl* (New York, N. Y.).

Avakumovic, Vojislav G. Sur l'équation différentielle de Thomas-Fermi. Acad. Serbe Sci. Publ. Inst. Math. **1**, 101–113 (1947).

The author considers the asymptotic behavior, for large x, of $o(1)$-solutions of $y''=f(x)y^\lambda$, in case $\lambda>1$ and $f(x)$ is a positive continuous function. The main result is that if $y(x)=o(1)$, $f(x)\sim x^vL(x)$, where $v>-2$, $L(tx)/L(x)\sim1$ for every fixed $t>0$ and $L(x)=o(x^\epsilon)$ for $\epsilon>0$, then $y(x)\sim p(x)$, where $x^{v+\lambda}L(x)p^{\lambda-1}(x)\sim(1+\lambda+v)(2+v)$. The proof proceeds by considering a suitable function $p(x)$, satisfying the last relation and $p''\sim f(x)p^\lambda(x)$, and applying a "variation of constants" $y(x)=p(x)z(\varphi(x))$, where φ is a solution of $p\varphi''+2p'\varphi'+\alpha p\varphi'^2=0$, $\alpha>0$. The resulting differential equation for z is of the form $z_{\varphi\varphi}-\alpha z_\varphi=\chi(x)z(\Omega(x)z^{\lambda-1}-1)$, where $\chi(x)\geqq\beta>0$ and $\Omega(x)\sim1$. Tauberian considerations show that $z(\varphi(x))\sim1$. Other theorems furnish estimates, instead of asymptotic formulae, for $y(x)$ under less restrictive assumptions on $f(x)$. *P. Hartman* (Baltimore, Md.).

Avakumovic, Vojislav G. Sur l'équation différentielle de Thomas-Fermi. II. Acad. Serbe Sci. Publ. Inst. Math. **2**, 223–235 (1948). (French. Serbian summary)

This paper is a continuation of the one reviewed above and deals with the differential equation for z, mentioned near the end of the preceding review. It is shown that, under suitable conditions, the asymptotic formula $z(\varphi(x))\sim1$ above can be refined to one of the form

$$z=1-\sum_1^{n-1}A_jc^j\exp\,(ja\varphi(x))+O(\exp\,na\varphi(x)).$$

This result is applied to obtain refined asymptotic formulae for $o(1)$-solutions of $y''=f(x)y^\lambda$. For example, an $o(1)$-solution of the Thomas-Fermi equation $y''=x^{-\frac12}y^{\frac32}$ satisfies $y(x)x^3/144=1-\sum_1^{n-1}A_jc^jx^{-j\tau}+O(x^{-n\tau})$, where $\tau=\frac12(7+\sqrt{73})$; $A_1, A_2, \cdots$ are absolute constants, c an integration constant, and n an arbitrarily fixed positive integer. *P. Hartman* (Baltimore, Md.).

Karamata, J. Sur l'application des théorèmes de nature tauberienne à l'étude des valeurs asymptotiques des équations différentielles. Acad. Serbe Sci. Publ. Inst. Math. **1**, 93–96 (1947).

It is shown that a simple Tauberian theorem of the author leads to $y(x)=o(x^{-\theta+\epsilon})$, as $x\rightarrow\infty$, for every $\epsilon>0$ if it is assumed that $y(x)$ is a solution of $y''=f(x)y^\lambda$; $\lambda>1$; $f(x)$ a positive continuous function satisfying $f(x)>cx^{-\delta}$, where $0<\delta<2$; $y(x)=o(1)$; and $\theta=(2-\delta)/(\lambda-1)$. (As pointed out by Karamata this is not as good as the result $y(x)=O(x^{-\theta})$, obtained in the first paper of Avakumovic reviewed above). *P. Hartman* (Baltimore, Md.).

Putnam, C. R. On the spectra of certain boundary value problems. Amer. J. Math. **71**, 109–111 (1949).

Let $q(x)$ be a real continuous function over $0\leqq x<\infty$, belonging to class L^2. Then the equation $y''+(\lambda+q(x))y=0$ is in the Grenzpunktfall and $\lambda\geqq0$ belongs to the spectrum associated with an arbitrary real homogeneous boundary condition at $x=0$. The author remarks that it remains undecided whether $\lambda\geqq0$ is in the continuous spectrum. *N. Levinson* (Cambridge, Mass.).

Hartman, Philip. On the spectra of slightly disturbed linear oscillators. Amer. J. Math. **71**, 71–79 (1949).

Let $q(t)$ be a real continuous function satisfying $q(t)\rightarrow0$ as $t\rightarrow\infty$. The author shows that the half-line $0\leqq\lambda<\infty$ is in the spectrum determined by $x''+(\lambda+q)x=0$ with a real homogeneous boundary condition at $t=0$ and the L^2 condition at $t=\infty$. The author observes that it remains undecided whether or not every $\lambda\geqq0$ is in the continuous spectrum but that it is in any case known that there may be positive characteristic values. *N. Levinson* (Cambridge, Mass.).

Hartman, Philip, and Wintner, Aurel. A criterion for the non-degeneracy of the wave equation. Amer. J. Math. **71**, 206–213 (1949).

Let $f(t)$ be a real continuous function and let $x(t)$ be a real solution of (*) $x''+f(t)x=0$. Let $f^+(t)=\max\,(f(t),0)$. If $\int^tf^+(s)ds=O(t^3)$ as $t\rightarrow\infty$, then there cannot be two independent solutions of (*) of class L^2 at infinity, i.e., (*) is in the limit point case. *N. Levinson* (Cambridge, Mass.).

Hartman, Philip, and Wintner, Aurel. On the location of spectra of wave equations. Amer. J. Math. **71**, 214–217 (1949).

The equation $x''+(f(s)+\lambda)x=0$ is considered for real continuous $f(s)$, where $\limsup f(s)<\infty$ as $s\rightarrow\infty$. If a solution of the equation for $\lambda=\lambda_0$ satisfies $x(s)=O(1)$ as $s\rightarrow\infty$ then either $x(s)$ belongs to L^2 at infinity or else $\lambda=\lambda_0$ is in the essential spectrum of the equation. *N. Levinson.*

Wintner, Aurel. Successive approximations and a property of the exponential series. Mat. Tidsskr. A. **1948**, 22–24 (1948).

The property that the error in representing e^{-t} by the first $n+1$ terms of its power series is positive or negative

according as $n=1, 3, 5, \cdots$ or $n=2, 4, 6, \cdots$ when $0\leqq t<\infty$ is extended to the positive monotone decreasing solutions of $x''+f(t)x=0$, $0\leqq t<\infty$, where $f(t)$ is continuous and negative. The series representation for $x(t)$ is obtained by a successive approximation procedure. The case e^{-t} corresponds to $f(t)\equiv -1$. *N. Levinson* (Cambridge, Mass.).

Wintner, Aurel. A criterion of oscillatory stability. Quart. Appl. Math. 7, 115–117 (1949).

Let $\int^t f(s)ds\rightarrow\infty$ as $t\rightarrow\infty$. If $N(t)$ is the number of zeros in the interval (t_0, t) of a solution $x(t)$ of $x''+f(t)x=0$, then $N(t)\rightarrow\infty$ as $t\rightarrow\infty$. *N. Levinson* (Cambridge, Mass.).

Borg, Göran. On a Liapounoff criterion of stability. Amer. J. Math. 71, 67–70 (1949).

Using an inequality due to Beurling, the author gives a very short proof of the following generalization of a theorem of Liapounoff. If $\phi(x)$ is continuous and of period π and if (1) $\int_0^\pi\phi(x)dx\geqq 0$, $\phi(x)\not\equiv 0$ and (2) $\int_0^\pi|\phi(x)|dx\leqq 4/\pi$ then the solutions of $y''+\phi(x)y=0$ are stable. It is also shown that the theorem is "best possible" in the sense that if either (1) or (2) is weakened the equation may be unstable. *N. Levinson* (Cambridge, Mass.).

Erugin, N. P. Generalization of a theorem of Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 632–638 (1948). (Russian)

The solutions of $x''+p(t)x=0$, where $p(t)$ is periodic of period 1 and $p(t)\geqq 0$, are stable if $\int_0^1 p(t)dt\leqq 4$ according to a well-known theorem of Liapounoff. By analyzing the proof of Liapounoff in greater detail the author derives other criteria for stability. *N. Levinson* (Cambridge, Mass.).

Letov, A. M. On a special case in the investigation of the stability of a system of regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 729–736 (1948). (Russian)

The system $dx_i/dt=\sum_{j=1}^n a_{ij}x_j+c_iy$, $i=1, 2, \cdots, N$; $dy/dt=f(z)$, $z=\sum_{i=1}^N l_ix_i-y$, is investigated, where $f(z)$ is continuous and bounded for all z, satisfying $zf(z)>0$, $z\neq 0$. By a series of transformations and a generalization of a method of Liapounoff, it is shown how the stability or instability of the solutions can be decided by certain expressions involving the coefficients a_{ij}, c_i, l_i. *R. Bellman* (Stanford University, Calif.).

Persidskiĭ, K. P. On the theory of stability of solutions of differential equations. Uspehi Matem. Nauk (N.S.) 1, 5-6(15-16), 250–255 (1946). (Russian)

This is a short account of a thesis; the following are typical results. (I) Consider the system (1) $\dot{x}=w(x, t)$, where x, w are n-vectors and w has continuous partial derivatives of the first order with respect to x, and $w(0, t)=0$. The author states that a necessary and sufficient condition for the stability of the solution $x=0$ is that there exists a positive definite function v such that its total derivative dv/dt is either negative or vanishes identically [the necessity of this condition is apparently not true; see Malkin, Sbornik Naučnyh Trudov Kazanskogo Aviacionnogo Instituta, no. 7 (1937), p. 22]. The following theorem on instability is proved. Let us call a region ω a "sector" when for any $\epsilon>0$ there exists an interior point $(a, 0)\epsilon\omega$, $0<|a|<\epsilon^2$, such that if $x=u(t)$ is the integral through $(a, 0)$, $|u|<\rho^2$ and $u\epsilon\omega$ for $t\geqq 0$. Then, the solution $x=0$ of (1) is unstable if the system is such that a sector ω and a function v exist such that (i) $|v|\leqq L<\infty$ in $\bar{\omega}$; (ii) $v>0$ if $x\neq 0$, x interior to ω; (iii) $dv/dt\geqq\eta(v, t)$ in the interior of ω, with $\eta\geqq 0$, $\eta(\alpha, t)$ nonincreasing with respect to α, $\int_0^\infty\eta(\alpha, t)dt=\infty$ for any $\alpha>0$.

(II) Suppose $w=P(t)\cdot x+L(x, t)$, where P is a matrix and L denotes terms of order higher than the first. Examples are given that show that conditions of Liapounoff for the stability of (1) based on the stability of the variational equations are not necessary. The author proves, furthermore, that if a positive definite function v exists, which has an infinitely small upper bound and such that dv/dt (calculated by means of the variational equations) is negative definite, then the solutions of (1) are uniformly stable.

(III) Let $f(t)$ be continuous for $t\geqq 0$; we say that f is of weak variation if, given $\epsilon>0$ and T as large as we please, there is an $N(\epsilon, T)$ such that, if $t', t''\geqq N$, $|t'-t''|<T$, then $|f(t')-f(t'')|<\epsilon$. Any function which has a finite limit for $t\rightarrow\infty$ or such that $f'(t)\rightarrow 0$ is of weak variation. Consider a system (2) $\dot{x}=P(t)\cdot x$, where the elements of P are bounded functions of weak variation; let $u_i(t)$ be the real parts of the characteristic roots of $P(t)$. Theorem: if $-u_i\geqq\alpha>0$, the characteristic numbers of (2) are not less than α and the solution $x=0$ of (1) (whose variational equations are (2)) is uniformly and asymptotically stable; if $u_i\geqq\beta>0$, the characteristic numbers do not exceed $-\beta$ and $x=0$ is unstable.

(IV) Let $Q(t)$ be the matrix of period τ such that $P=Q$ if $0\leqq t\leqq\tau$. Then a necessary and sufficient condition for the system (2) to be regular [Liapounoff, Problème Général de la Stabilité du Mouvement, Princeton University Press, 1947, p. 236; these Rev. 9, 34] is that the characteristic numbers of $\dot{x}=Q(t)\cdot x$ have limits as $\tau\rightarrow\infty$; these limits are then the characteristic numbers of (2). This theorem enables the author to extend classical theorems valid for periodic systems to regular systems. For example, if the system is regular and canonical its characteristic numbers $\alpha_1\geqq\cdots\geqq\alpha_{2n}$ satisfy $\alpha_s+\alpha_{2n-s+1}=0$. *J. L. Massera* (Montevideo).

Bautin, N. N. Criteria for unsafe and safe bounds of a region of stability. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 691–728 (1948). (Russian)

The author treats first the problem of stability and instability of the solutions of

$$(1)\qquad dx_i/dt=\sum_{j=1}^{3}a_{ij}x_j+P_i(x_1, x_2, x_3),\qquad i=1, 2, 3,$$

where the P_i are power series in the x_i lacking constant and first degree terms. Although a reasonably complete theoretical solution to the problem has been given by Liapounoff and his pupils, the algebraic-transcendental problem of determining the stable and unstable regions of coefficient space remains. The simplest regions, corresponding to characteristic roots with negative real parts, may be determined quite easily by applying Hurwitz's criteria to the characteristic equation of $A=(a_{ij})$. If there are characteristic roots with zero real parts, the coefficients of the terms in the P_i play an influential role. The author presents the results of the necessary calculations for these cases and applies the results to the treatment of several electronic circuits. After the discussion of the three-dimensional case, the four-dimensional case is attacked. Here the calculations are truly Herculean in magnitude. There are several clear geometric diagrams complementing the text.

R. Bellman (Stanford University, Calif.).

Cafiero, Federico. Sui teoremi di unicità relativi ad un'equazione differenziale ordinaria del primo ordine. Giorn. Mat. Battaglini (4) 2(78), 10–41 (1948).

Under varying hypotheses this paper proves five uniqueness theorems. The last of these, which contains and generalizes the known criteria, may be stated as follows. The function $f(x, y)$ is defined in the set $C: x_0 \leqq x \leqq x+a, y \epsilon E(x)$, where $E(x)$ is a number set varying with x. The analysis employs a function $F(x, u, P)$ which is defined for every point P with coördinates (ξ, η) in C and which, in addition to satisfying Carathéodory's hypotheses [Vorlesungen über reelle Funktionen, 2d ed., Teubner, Leipzig, 1927, p. 665] in every set of the type $x_0 < x_0+\beta_1 \leqq x \leqq x_0+a$, $0<\beta_2 \leqq u < \infty$, also satisfies the following: (i) $f(x, y_1)-f(x, y_2) \leqq F(x, y_1-y_2, P)$ for almost all x in a neighborhood to the right of ξ and for y_1, y_2 in a neighborhood of η and satisfying $y_2<y_1$; (ii) the set of x's in C is the sum of two sets T_1 and T_2 such that there is a positive h making the upper integral of $u=hk+\int_{\xi+\delta}^{x} F(t, u, P)dt$ less than or equal to a prescribed positive ϵ for $\xi+\delta<x$ and for every k, δ satisfying $0<k=\delta \leqq h$, if ξ is in T_1, and $k=1$, $0<\delta<h$, if ξ is in T_2; (iii) $f(x, y)$ is uniformly continuous in y in a neighborhood of each point of C whose x is in T_1. Under these conditions an absolutely continuous solution of $y'=f(x, y)$ is uniquely determined to the right of the initial value of x by the initial value of y.

J. M. Thomas (Durham, N. C.).

Haag, Jules. Sur l'approximation des solutions associées d'un système différentiel à coefficients périodiques. Bull. Sci. Math. (2) 72, 69–72 (1948).

The author considers the system of n differential equations $x_i'=\lambda f_i(x, t)$, where λ is a parameter and f_i is of period T in t, and the associated system $X_i'=\lambda F_i(X)$, where $F_i(X)=T^{-1}\int_0^T f_i(X, t)dt$. Let $x=x(t)$ and $X=X(t)$ be solutions of these systems satisfying the same initial condition at $t=0$; and let s be a fixed arc-length on $X=X(t)$, from $t=0$ to $t=t(s)$. The author shows that the distance $|x(t(s))-X(t(s))|$ is $O(\lambda)$, as $\lambda \to 0$, under suitable smoothness conditions on f_i [allowing some discontinuities as in his paper in the same Bull. (2) 71, 205–219 (1947); these Rev. 10, 195].

P. Hartman (Baltimore, Md.).

Malkin, I. G. The oscillations of systems with one degree of freedom, close to systems of Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 561–596 (1948). (Russian)

The author observes that with the usual analytical methods which consider nonlinear equations as quasilinear (i.e., the coefficients of the nonlinear terms contain a small parameter) it is not possible in general to obtain the totality of periodic solutions. He studies systems (1) $\dot{x}=-\lambda y+X(x,y)+\mu f(t,x,y,\mu), \dot{y}=\lambda x+Y(x,y)+\mu F(t,x,y,\mu)$, where $X=-\partial H/\partial y$, $Y=\partial H/\partial x$ do not contain linear terms and f, F are periodic in t of period 2π; all these functions are supposed to be analytic. The nonlinear canonical system (2) $\dot{x}_0=-\lambda y_0+X(x_0, y_0)$, $\dot{y}_0=\lambda x_0+Y(x_0, y_0)$ is called the generator system of (1). The application of classical methods to (2) yields the existence (provided λ satisfies certain restrictions) of analytic solutions $[x_0^{(n)}, y_0^{(n)}]$ periodic with period $2\pi/n$, n an integer. These "generator solutions" are the starting point for the calculation of periodic solutions of (1), analytic in μ: (3) $x^{(n)}=x_0^{(n)}(t-\alpha)+\mu x_1(t)+\cdots$, $y^{(n)}=y_0^{(n)}(t-\alpha)+\mu y_1(t)+\cdots$. Application of Poincaré's methods gives the proof of the following theorem. For the existence of a periodic solution (3) of (1) it is necessary that α be a root and sufficient that α be a simple root of the equation

$$\int_0^{2\pi} [f_0 \cdot dy_0^{(n)}(t-\alpha)/dt - F_0 \cdot dx_0^{(n)}(t-\alpha)/dt]dt=0,$$

where $f_0=f[t, x_0^{(n)}(t-\alpha), y_0^{(n)}(t-\alpha), 0]$, and F_0 is defined similarly. The practical calculation of the series follows the usual procedure of indeterminate coefficients. A similar theorem is proved for the existence of a solution (x^0, y^0) of (1) which tends to 0 as $\mu \to 0$.

The author considers next the "resonance" cases, where $\lambda=n+\mu a$; the previous method is not valid in this case. Assuming certain restrictions on the Fourier coefficients of $f(t, 0, 0, 0)$ and $F(t, 0, 0, 0)$, he is able to prove that a periodic solution (x_*, y_*) of (1) exists which tends to 0 as $\mu \to 0$ and which is analytic in $\nu=\mu^\omega$, $\omega=1/(2s+1)$, s being an integer depending on the form of the solutions of (2).

After discussing stability questions, the author applies his methods to Duffing's equation:

$$\ddot{x}+k^2x-\gamma x^3=\mu(a \cos pt+b \cos qt-2h\dot{x}),$$

γ, μ, h positive; p, q integers. If q/p is not an odd integer, two real solutions $x^{(p)}$ exist when $p<k$. Several terms of the series developments of these solutions, as well as of x^0 and x_* (when $k^2=p^2-\mu\lambda$) are calculated and the stability properties of these solutions are discussed in great detail with varying k.

J. L. Massera (Montevideo).

Malkin, I. G. Oscillations of systems with several degrees of freedom, close to systems of Lyapunov. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 673–690 (1948). (Russian)

This is the extension to more degrees of freedom of the paper reviewed above. The assumptions on the system are such that under a suitable real affine transformation of coordinates it takes the form

$$dx/dt=-\lambda y+X+\mu f, \quad dy/dt=\lambda x+Y+\mu F,$$
$$dx_s/dt=\sum r_{si}x_i+X_s+\mu f_s, \qquad s=1, \cdots, m,$$

where all the functions are analytic in the coordinates near the origin, X, Y and X_s contain only the coordinates and these to powers not less than 2, while f, F and the f_s contain in addition μ and t. Moreover, the constant matrix $\|r_{si}\|$ has no pure complex or zero characteristic roots. Finally it is assumed that for $\mu=0$ there is a general integral of the form $H=x^2+y^2+S(x, y, x_1, \cdots, x_m)=$constant, where H is analytic at the origin and contains no linear terms, while the quadratic terms in S do not contain x or y. Under these assumptions the author succeeds in extending the results of his preceding paper to the more general situation.

S. Lefschetz (Princeton, N. J.).

Levinson, Norman. Transformation theory of non-linear differential equations of the second order. Ann. of Math. (2) 49, 738 (1948).

Corrections are made to an earlier paper [same Ann. (2) 45, 723–737 (1944); these Rev. 6, 173]. An additional hypothesis is inserted to guarantee existence of a periodic solution and a stability criterion is corrected.

W. Kaplan (Ann Arbor, Mich.).

Minorsky, Nicolas. Sur l'oscillateur de van der Pol. C. R. Acad. Sci. Paris 228, 60–61 (1949).

The author makes qualitative observations based on transforming the van der Pol equation by means of polar coordinates in the phase plane.

N. Levinson.

Carrier, G. F. A note on the vibrating string. Quart. Appl. Math. 7, 97–101 (1949).

The basic result of a previous paper [same Quart. 3, 157–165 (1945); these Rev. 7, 13] is obtained in a mathematically better way and a refinement of the solution with periodic motion of the lowest frequency is obtained.

N. Levinson (Cambridge, Mass.).

Reeb, Georges. Sur les singularités d'une forme de Pfaff analytique complètement intégrable. C. R. Acad. Sci. Paris 227, 1201–1203 (1948).

The author considers a Pfaffian form ω in complex n-space C^n. He assumes: ω can be expanded at the origin in a series $\omega=\omega_1+\omega_2+\cdots$, where the coefficients of ω_i are homogeneous polynomials of degree i; ω is completely integrable; $n\geqq 3$ and $\omega_1=\sum x_i dx_i$. He studies the integrals of ω by means of the map $\psi(\rho, \vec{u})=\rho\cdot\vec{u}$ of $C^1\times\Sigma^{n-1}$ into C^n, where Σ^{n-1} is the real unit sphere in C^n; in $C^1\times\Sigma^{n-1}$ the manifold $\rho=0$ is a compact, simply connected integral manifold, so that previous results of the author on laminated manifolds apply [same C. R. 224, 1613–1614 (1947); these Rev. 8, 595]. One result is that ω has an integral whose series expansion begins with $\sum x_i^2$; for real analytic forms a similar result is obtained, involving of course an arbitrary signature.

H. Samelson (Ann Arbor, Mich.).

Zervos, P. Sur le degré d'indétermination dans la théorie des équations différentielles. Bull. Math. Phys. Éc. Polytech. Bucarest 10 (1938–39), 3–13 (1940).

This lecture discusses the question: what supplementary (boundary or initial) conditions are to be adjoined to a system of differential equations in order to render the solution unique? The general remarks made by the author are illustrated by specific results, principally about systems of equations of Monge's type and the so-called "special" type of Pfaffian system, drawn for the most part from papers by Goursat and Cartan. *J. M. Thomas* (Durham, N. C.).

Kosambi, D. D. Systems of partial differential equations of the second order. Quart. J. Math., Oxford Ser. 19, 204–219 (1948).

Attacher à un système différentiel des propriétés indépendantes de certains choix des variables, c'est essentiellement l'objet de la géométrie des "paths" de l'école de Princeton. Le présent travail étudie, de ce point de vue, les systèmes aux dérivées partielles du second ordre résolus par rapport à toutes les dérivées du second ordre de n fonctions inconnues (x) des m variables indépendantes (t), le système étant supposé invariant pour les groupes de transformations fonctionelles généraux transformant d'une part les x, d'autre part les t. Après avoir formé les conditions d'invariance (et l'intégrabilité), l'auteur indique comment on peut déterminer les invariants différentiels. Il recherche ensuite à quelles conditions un tel système peut être rattaché à un problème convenable de "variation." *M. Janet* (Paris).

Samarskiĭ, A. On the influence of constraints on the characteristic frequencies of closed volumes. Doklady Akad. Nauk SSSR (N.S.) 63, 631–634 (1948). (Russian)

Let T be a bounded domain in n $(\geqq 2)$ dimensions, with a suitably smooth boundary Γ, and E a closed set lying in T. Consider the eigenvalue problem $\Delta v+\lambda v=0$ on T, $v=0$ on Γ, $v=0$ on E (constraint condition). The author is concerned with the variation of the eigenvalues and eigenfunctions of the unconstrained eigenvalue problem (i.e., when E is the empty set and the constraint condition is superfluous) produced by the introduction of the constraint condition (i.e., by taking E not empty), and with the dependence of this variation on the structure of the set E. The structure of E is characterized by its capacity with respect to Γ [N. Wiener, J. Math. Physics 3, 24–51 (1924)]. Theorem 1. If v satisfies $\Delta v+\lambda v=0$ on $T-E$ and is bounded in a neighborhood of a set E of capacity zero, then v is differentiable on E and satisfies the equation on E. Corollary. The unconstrained eigenfunction with lowest eigenvalue is not a constrained eigenfunction. Theorem 2. If $\Delta v+\lambda v=0$ on T then the set of all zeros of v in T cannot have capacity zero. Another theorem gives an estimate of the variation of the lowest unconstrained eigenvalue in terms of the maximum of the first unconstrained eigenfunction on E and the capacity of E with respect to Γ. No proofs are given. *J. B. Dias.*

Foà, Emanuele. Sulla trasmissione del calore in mezzi isotropi o anisotropi con coefficiente di conduttività variabile con la temperatura. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 4 (1946–47), 119–122 (1948).

In certain media the temperature u satisfies an equation of the type

$$(*)\qquad \sum_{i=1}^{3} n_i\frac{\partial}{\partial x_i}\left(f(u)\frac{\partial u}{\partial x_i}\right)=c\rho\frac{\partial u}{\partial t},$$

where n_i are constants. Placing $\theta=\int_0^u f(u)du$, $\xi_i=x_i n_i^{-\frac{1}{2}}$, the author points out that equation (*) takes the form

$$\sum_{i=1}^{3}\frac{\partial^2\theta}{\partial\xi_i^2}=\frac{c\rho}{f(u)}\frac{\partial\theta}{\partial t}.$$

F. G. Dressel (Durham, N. C.).

Ludwig, K. Das Aufheizen einer Wand durch eine anlaufende Heizanlage. Ing.-Arch. 16, 45–50 (1947).

The method of particular solutions is used to find the function $u(x, t)$ which satisfies the heat equation $u_t=au_{xx}$, is zero for $t=0$, and satisfies the boundary conditions $u_x(0, t)=q(1-e^{-\alpha t})$, $u_x(b, t)=cu(b, t)$, where a, b, c, α, q are constants. The problem is also treated by the use of the Laplace transform. *F. G. Dressel* (Durham, N. C.).

Nordon, Jean. Une solution nouvelle de l'équation de la chaleur à $n+1$ variables. C. R. Acad. Sci. Paris 228, 459–460 (1949).

Let (*) $g^{ij}(\partial^2 u/\partial x^i\partial x^j-\Gamma_{ij}\partial u/\partial x_k)=\partial u/\partial t$ be the heat equation associated with the linear element $ds^2=g_{ij}dx^i dx^j$ $(i, j=1, \cdots, n)$. The author states sufficient conditions on the space so that a solution of (*) can be represented in the form (**) $u=\varphi(t)f(z)$, $z=\rho(x_1, \cdots, x_n)/\psi(t)$. For a Euclidean space in which r represents geodesic distance, solutions of the type (**) are written down taking $4\rho=r^2$, $\psi=t$, $\varphi=t^{k-n/4}$, where k is an arbitrary constant. *F. G. Dressel.*

Rubinšteĭn, L. I. On the stability of the boundary of the phases in a two-phase heat-conducting medium. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. 12, 557–560 (1948). (Russian)

It is shown that in the two-phase linear heat-conduction problem of Stefan the position of the boundary is a continuous function of the boundary values and also of the thermal constants which characterize the conducting medium. The proof applies to the case when at the initial moment both the liquid and solid phases exist, and is based on the proof for the existence and the uniqueness of the solution of the

problem which was given by the author in earlier papers [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 11, 37–54 (1947); Doklady Akad. Nauk SSSR (N.S.) 58, 217–220 (1947); these Rev. 8, 516; 9, 287].

H. P. Thielman (Ames, Iowa).

Polubarinova-Kočina, P. Ya. On a nonlinear partial differential equation arising in the theory of filtration. Doklady Akad. Nauk SSSR (N.S.) 63, 623–626 (1948). (Russian)

In suitable coordinates, one of the boundary value problems considered is that of (*) $\partial^2 u^\mu/\partial x^2 = \partial u/\partial \tau$, $\mu \geqq 1$, subject to $u(0, \tau) = 1$, $u(x, 0)$ a given constant. This is treated by setting $\xi = \frac{1}{2} x \tau^{-\frac{1}{2}}$, which transforms (*) into $d^2 u^\mu/d\xi^2 = -2\xi du/d\xi$. The solution of this equation, subject to the conditions $u = 1$, $du/d\xi = \alpha$, for $\xi = 0$, where α is a real number, is such that $\lim_{\xi \to +\infty} u(\xi)$ exists, thus yielding a solution of the original problem. The solution is carried out numerically for the physically interesting case $\mu = 2$, and compared with the linear approximation $\mu = 1$. The more complicated case when $u(0, \tau) = 0$ is also treated.

J. B. Diaz (Providence, R. I.).

Ingersoll, Benham M. An initial value problem for hyperbolic differential equations. Bull. Amer. Math. Soc. 54, 1117–1124 (1948).

Given the linear differential equation of hyperbolic type, $L(u) = u_{xy} + au_x + bu_y + cu = d$, where a, b, c, d are given functions of (x, y), a classical problem is to determine an integral $u(x, y)$ which assumes prescribed values on two characteristics passing through a point: for example, one can assign the functions $u(x, 0)$, $u(0, y)$. The author supposes that the values of the derivatives $\partial^k u(x, y)/\partial x^k|_{x=0}$ and $\partial^m u(x, y)/\partial y^m|_{y=0}$ are prescribed and shows that this generalized problem still has one and only one solution (under suitable hypotheses).

L. Amerio (Genoa).

Bureau, Florent. La solution élémentaire d'une équation linéaire aux dérivées partielles, décomposable et totalement hyperbolique, d'ordre quatre et à quatre variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 566–592 (1948).

Let (1) $f(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial t)v = g(x_1, x_2, x_3, t)$ be a totally hyperbolic differential equation with constant coefficients, where f is a form of degree 4. In a previous paper [same Bull. Cl. Sci. (5) 33, 684–711, 827–853 (1947); these Rev. 9, 513] the author gave an expression for the elementary solution $v(x_1, x_2, x_3, t)$ of (1) in the form of a double integral extended over the characteristic surface $f(\xi_1, \xi_2, \xi_3, 1) = 0$. In the present paper, which is concerned with the special case where f is the product of two quadratic forms, the author succeeds in reducing v to simple integrals. The two ellipsoids forming the characteristic surface in this case are assumed to have no real intersections. Their equations can be taken in the form $\sum_{i=1}^{3} \xi_i^2 = 1$ and $\sum_{i=1}^{3} e_i \xi_i^2 = e_4$, where $0 < e_1 < e_2 < e_3 < e_4$. By a rather complicated argument, which involves the elliptic integrals attached to the imaginary curve of intersection of the two ellipsoids, the author arrives at the following expressions for $v(x_1, x_2, x_3, t)$:

$$\tfrac{1}{2}\pi \int_{-\infty}^{0} \log (t^2 - \Lambda) \frac{d\lambda}{\sqrt{P(\lambda)}}, \qquad t^2 - e_4 g > 0,$$

$$\int_{e_4}^{\infty} \log (t^2 - \Lambda) \frac{d\lambda}{\sqrt{P(\lambda)}}, \qquad t^2 - f > 0 > t^2 - e_4 g,$$

$$0, \qquad 0 > t^2 - f.$$

Here

$$f = \sum_{i=1}^{3} x_i^2, \quad g = \sum_{i=1}^{3} x_i^2/e_i, \quad P(\lambda) = \prod_{i=1} (\lambda - e_i),$$

$$\Lambda = (\lambda - e_4) \sum_{i=1}^{3} x_i^2/(\lambda - e_i)$$

F. John (New York, N. Y.).

Integral Equations

Parodi, Maurice. Sur une conséquence des propriétés de l'équation intégrale de Schlömlich. Bull. Sci. Math. (2) 72, 7–8 (1948).

The author reduces the integral equation

$$\varphi(p) = (2/\pi) \int_0^{\pi/2} \psi(p/\sin \theta) d\theta,$$

by operational methods, to Schlömilch's integral equation

$$f(x) = (2/\pi) \int_0^{\pi/2} \Phi(x \sin \theta) d\theta.$$

[The reviewer remarks that the reduction is effected more simply by the substitution $p = 1/x$, $\psi(y) = \Phi(1/y)$, $\varphi(1/x) = f(x)$.]

A. Erdélyi (Pasadena, Calif.).

Parodi, Maurice. Sur la détermination d'une famille de noyaux réciproques. Bull. Sci. Math. (2) 72, 66–68 (1948).

Cf. the author's note in C. R. Acad. Sci. Paris 226, 1877–1878 (1948); these Rev. 10, 36.

A. Erdélyi.

Thielman, H. P. On a class of singular integral equations occurring in physics. Quart. Appl. Math. 6, 443–448 (1949).

The author remarks that if the function $K(|x-y|)$ satisfies the same linear homogeneous differential equation with constant coefficients for $0 \leqq y \leqq x$ or $0 \leqq x \leqq y$, then the integral equations of the Wiener-Hopf type,

(A) $$f(x) = \int_0^\infty K(|x-y|) g(y) dy$$

or

(B) $$f(x) = g(x) - \lambda \int_0^\infty K(|x-y|) g(y) dy$$

may be solved by forming a differential equation for $g(x)$. Upon substituting the function $g(x)$ found for either equations (A) or (B) into the appropriate equation, the arbitrary constants may be determined. For equation (A) with $K(|x|) = e^{-|x|}$, there are special conditions on $f(x)$. [The determination of the constants is an important issue in the solution of this particular class of integral equations. The Fourier transform method does this all at once, thereby eliminating much tedious algebra.]

A. E. Heins.

Ahiezer, N. I. Integral operators with Carleman kernels. Uspehi Matem. Nauk (N.S.) 2, no. 5(21), 93–132 (1947). (Russian)

This study relates to symmetric kernels $K(s, t)$. When $\iint |K(s, t)|^2 ds dt < \infty$ $(a \leqq s, t \leqq b)$, the operator

$$Af = \int_a^b K(s, t) f(t) dt$$

in the space $L^2(a, b)$ is completely continuous and $K(s, t)$ is a Hilbert-Schmidt kernel. The case when

$$\int\left|\int K(s, t)f(t)dt\right|^2 ds \leqq M^2 \int |f^2(t)|dt$$

(all f in L^2) is the one extensively studied by Hilbert; A is then a bounded operator. Still more general is the case when $\int_a^b |K(s, t)|^2 dt < \infty$ for almost all s. Integral equations with kernels of the latter kind (as well as some more general kernels) were studied by T. Carleman. The author presents an exposition of an essential part of Carleman's theory; the developments are given in the garb of modern Hilbert space theory [following A. Plesner, Uspehi Matem. Nauk 9, 3–125 (1941); these Rev. 3, 210]. The terminology is largely that of M. H. Stone [Linear Transformations in Hilbert Space and their Applications to Analysis, Amer. Math. Soc. Colloquium Publ., v. 15, New York, 1932].

Chapter 1 [pp. 94–106] presents some general facts relating to symmetric operators. In chapter 2 [pp. 106–115] there is given a study of Carleman's integral operators. We denote by L^2 the space of $f(x)$ $(-\infty < x < \infty)$ (the results can be extended to any n-dimensional point-space of positive Lebesgue measure). A Carleman kernel is a measurable (possibly complex-valued) function $K(s, t)$ $(-\infty < s, t < \infty)$ such that: (a) $\overline{K(s, t)} = K(t, s)$ (almost everywhere in the plane), (b) $\int_{-\infty}^{\infty} |K(s, t)|^2 dt < \infty$ (almost all s). In chapter 3 [pp. 115–128] a study is presented of the spectral functions of integral operators with Carleman kernels (representation of any bounded operator in L^2; properties of Carleman spectral functions; integral representations of Carleman kernels; representation for $\int K(s, t)f(t)dt$). Chapter 4 [pp. 128–132] treats the following unsolved problem: to find conditions for a given symmetric operator T in $L^2(-\infty, \infty)$ to be an integral operator with a Carleman kernel [see, for example, J. von Neumann, Charakterisierung des Spektrums eines Integraloperators, Actual. Sci. Ind., no. 229, Hermann, Paris, 1935]. The author gives a solution of this problem for the particular class of Carleman-B kernels, for which $|K(s, t)| \leqq M(s)M(t)$ for almost all s and t ($M(s) \geqq 0$ being finite, measurable). It is proved that in this case it is necessary and sufficient that there should exist a finite measurable $P(s)$ (almost everywhere positive) so that: $[L^2]_P \subset D_{T^*}$, $|(T^*f, g)| \leqq \|f\|_P \cdot \|g\|_P$, $|(T^*f, T^*g)| \leqq \|f\|_P \cdot \|g\|_P$, $(T^*f, g) = (g, T^*g)$ for all f, g in $[L^2]_P$.

W. J. Trjitzinsky (Urbana, Ill.).

Vaĭnberg, M. M. The existence of solutions of a system of nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) 63, 605–608 (1948). (Russian)

The author studies the system

$$(1)\qquad \mu_i u_i(x) = \int_B K_i(x, y) g_i(u(y), y) dy, \qquad i = 1, \cdots, n;$$

here

$$u = (u_1, \cdots, u_n),\ g_i(u, x) = (\partial/\partial u_i) G(u, x),\ g_i(0, x) = G(u, x) = 0;$$

B is a bounded domain in a Euclidean space. The set of functions $(\psi_1(x), \cdots, \psi_n(x))$, considered as an element of the space $L_{2,n}(B)$ of norm not equal to 0, is termed a solution of (1) if it satisfies (1) for some real values of the μ_i. The element 0, satisfying (1) for all μ_i, is not considered to be a solution. It is proved that there exists a sequence of solutions of (1), tending to 0, if the $K_i(x, y)$ are symmetric, positive, bounded, measurable, while the real functions of real arguments $g_i(u, y)$ are (in a neighborhood of $u = 0$) measurable over B in y, continuous in u and $|g_i(u, y)| \leqq a_i(y)$, where the $a_i(y)$ are in L_2. It is also proved that there exists a system of solutions of (1), tending to 0 (according to the norm), if the $K_i(x, y)$ are positive, symmetric, $0 < \int_B \int_B |K_1(x, y)|^{2+\beta} dx dy < \infty$ $(\beta > 0)$, while the real $g_i(u, y)$ are continuous in u (real u), are measurable over B in y and $|g_i(u, y)| \leqq a_i(y) + b_i \sum_{k=1}^{n} |u_k|$ ($a_i(y)$ in L_2, $a_i(y) \geqq 0$, $b_i \geqq 0$).

W. J. Trjitzinsky (Urbana, Ill.).

Weinberg, A. M., and Schweinler, H. C. Theory of oscillating absorber in a chain reactor. Physical Rev. (2) 74, 851–863 (1948).

Under stationary conditions the pile equation is

$$(1)\qquad D\nabla^2 n_0(\mathbf{r}) - \sigma_p n_0(\mathbf{r}) + kp^{-1}\sigma_p(1 - \textstyle\sum \beta_j) \displaystyle\int_{-\infty}^{\infty} n_0(\mathbf{r}') P(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' + \sum_j \int_{-\infty}^{\infty} \frac{c_{j0}(\mathbf{r}')}{\tau_j} P_j(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' = 0$$

and (1′) $(k/p)\beta_j \sigma_p n_0(\mathbf{r}) = c_{j0}(\mathbf{r})/\tau_j$, where D, σ_p, k, p, β_j and τ_j are some assigned constants and $P(|\mathbf{r} - \mathbf{r}'|)$ and $P_j(|\mathbf{r} - \mathbf{r}'|)$ are known functions of the argument. By expressing $n_0(\mathbf{r})$ as a Fourier integral, $\int_{-\infty}^{\infty} \delta(\kappa_0^2) e^{i\kappa \cdot \mathbf{r}} d\mathbf{r}$ (δ is Dirac's δ-function), we have

$$\int_{-\infty}^{\infty} n_0(\mathbf{r}') P(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' = \bar{P}(\kappa_0^2) n_0(\mathbf{r}),$$

and

$$(2)\qquad \int_{-\infty}^{\infty} c_{j0}(\mathbf{r}) P_j(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' = \bar{P}_j(\kappa_0^2) n_0(\mathbf{r}),$$

where $\bar{P}(\kappa_0^2)$ and $\bar{P}_j(\kappa_0^2)$ are the three-dimensional Fourier transforms of $P(|\mathbf{r} - \mathbf{r}'|)$ and $P_j(|\mathbf{r} - \mathbf{r}'|)$, respectively. The characteristic equation satisfied by κ_0^2 is obtained by substituting the expressions (2) in (1).

If we now suppose that an absorber whose cross section at position $\mathbf{r}$ and time t is $\sigma_a(\mathbf{r}, t)$ is placed in the pile, the pile would not, on the average, operate under critical conditions unless the "multiplication constant" k is changed to a new value k'. After this adjustment, the pile equation is

$$(3)\qquad D\nabla^2 n(\mathbf{r}, t) - [\sigma_p + \sigma_a(\mathbf{r}, t)] n(\mathbf{r}, t) + k' p^{-1} \sigma_p (1 - \textstyle\sum \beta_j) \displaystyle\int n(\mathbf{r}', t) P(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' + \sum \int \tau_j^{-1} c_j(\mathbf{r}', t) P_j(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' = v^{-1} \partial n/\partial t,$$

and (4) $(k'/p)\beta_j \sigma_p n = c_j/\tau_j + \partial c_j/\partial t$ (v is to be regarded as another constant). The problem is to determine the difference between k' and k. This relation is found in the following manner. Multiplying (3) by $n_0(\mathbf{r})$ and (1) by $n(\mathbf{r})$, integrating over all space, reducing the integrals by Green's theorem, integrating over t from 0 to T, and finally passing to the limit $T = \infty$, we find that

$$(5)\qquad [(1 - \textstyle\sum \beta_j) \bar{P}(\kappa_0^2) + \sum \beta_j P_j(\kappa_0^2)] \sigma_p (k' - k) \displaystyle\int_{\text{pile}} \langle n n_0 \rangle_{\text{av}} d\mathbf{r} = p \int_{\text{pile}} \langle \sigma_a n n_0 \rangle_{\text{av}} d\mathbf{r},$$

where

$$(6)\qquad \langle \sigma_a n n_0 \rangle_{\text{av}} = \lim_{T \to \infty} T^{-1} \int_0^T \sigma_a n n_0 d\mathbf{r},$$

and $\langle n n_0 \rangle_{\text{av}}$ is similarly defined.

With the value of k' determined according to equation (5), we can return to equation (3) and ask for the more detailed

characteristics of the solution $n(\mathbf{r}, t)$ for special forms $\sigma_a(\mathbf{r}, t)$. For example, for a small absorber of volume V_a, oscillating with a frequency $2\pi/\omega$, at $\mathbf{r}_0$ with a small amplitude ρ, $\sigma_a(\mathbf{r}, t) = V_a\delta(\mathbf{r}-\mathbf{r}_0+\rho e^{i\omega t})$, where δ denotes Dirac's δ-function. Similarly, for an absorber of fluctuating strength, distributed uniformly through the pile,

$$\sigma_a(\mathbf{r}, t) = c(1+\alpha e^{i\omega t}) \quad (\alpha<1;\ c=\text{constant}).$$

The solution for these cases can be found in a straightforward fashion by expanding the solutions in a complete set of orthogonal functions $Z_\nu(r)$ determined as solutions of the equation $\nabla^2 Z_\nu(\mathbf{r})+\kappa_\nu^2 Z_\nu(\mathbf{r})=0$, which vanish on the boundary of the pile. *S. Chandrasekhar.*

Carafa, Mario. Calcolo del nucleo risolvente delle equazioni funzionali lineari, mediante un numero finito di integrazioni. Collectanea Math. 1, 1–62 (1948).

This is a detailed exposition of results published in summary form [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 521–527 (1947); these Rev. 9, 148]. The objective of the author is to reduce solutions of functional equations in analytic functions to a finite number of derivatives and integrations involving a finite number of "universal" functions. The basic idea is illustrated in the problem of determining the analytic continuation of $f(x)=\sum_n a_n x^n$ with radius of convergence r around the point x_1 with $|x_1|<r$. By introducing the universal function

$$P_{x_1}(x, t, \theta) = \sum_{k=0}^{\infty} \frac{(x-x_1)^k}{(t-x_1)^{k+1}} \frac{\theta^{k!}}{k!}$$

it is easy to show that

$$\sum_{n=0}^{\infty} f^{(n)}(x_1)(x-x_1)^n/n! = \frac{\partial}{\partial\theta}\bigg|_{\theta=1} \frac{1}{2\pi i}\int_C P_{x_1}(x, t, \theta) f(t)dt,$$

where C lies interior to $|x|=r$. More complicated is the expression for the resolvent of the linear functional equation

$$y(x) - \frac{\lambda}{2\pi i}\int_C \alpha^{-1} w(x, 1/\alpha) y(\alpha) d\alpha = f(x),$$

where $w(x, \alpha)=\sum_{k,h} s_{kh} x^k \alpha^h$ is convergent for $|x|<\rho_1$, $|\alpha|<\rho_2$ with $\rho_1\rho_2>1$ and C a circle of radius R with $\rho_1>R>1/\rho_2$. The expression for the resolvent takes the form:

$$\frac{\partial}{\partial\theta}\bigg|_{\theta=1} \frac{1}{(2\pi i)^4}\int_{C_r} dr \int_{C_\xi} d\xi \int_{C_t} dt \int_{C_u} du \frac{P_0(r, \theta) S(x, \alpha, \xi, r, t, u)}{\lambda\xi(1-u\lambda\bar{w}(\xi, r, t))},$$

where $P_0(r, \theta) = P_{x_1}(x, t, \theta)\,|_{x_1=0,\, x=r,\, t=1}$, S is a universal function, a power series in x, α, ξ, r, t, u, and $\bar{w}$ is the integral of the product of another universal function with w.

T. H. Hildebrandt (Ann Arbor, Mich.).

Functional Analysis, Ergodic Theory

Weyl, Hermann. Almost periodic invariant vector sets in a metric vector space. Amer. J. Math. 71, 178–205 (1949).

The Schmidt-Peter-Weyl integral equation method is put to work in very general circumstances, such as result from an axiomatic consideration of the method. The author considers (1) a linear space Σ with inner product (f, g) where besides the norm $\|f\| = (f, f)^{\frac{1}{2}}$, a length $|f| \geqq \|f\|$ is defined with respect to which Σ is complete; (2) a group σ of linear (doubly) isometric transformations $f\to sf$ of Σ onto itself. From a vector f in Σ which is almost periodic with respect to σ and the length (i.e., $\{|sf|\ s\epsilon\sigma\}$ is totally bounded) he forms the closure, in length, $\Sigma_0{}^f$ of all finite sums $\sum_{k=1}^{n} s_k f$. The problem is to find in $\Sigma_0{}^f$ a "complete" system of finite-dimensional, primitive invariant subspaces S. Such spaces S transform according to irreducible unitary representations Ω of σ (determined up to equivalence). The integral equation method works as usual (since mean values also exist for vector-valued almost periodic functions; a slightly new touch appears, however, in the establishing of certain inequalities). It delivers a denumerable number of representation classes Ω_n (degree h_n), $n=1, 2, \cdots$, of σ and (in the above sense) corresponding mutually orthogonal h_n-dimensional, primitive invariant subspaces $S_n^{(j)}$, $j\leqq m_n$ $(\leqq h_n)$, $n=1, 2, \cdots$, of $\Sigma_0{}^f$ with the following property. Let $g_1, g_2, \cdots$ be a full system of orthogonal unit vectors for all $S_n^{(j)}$; then for every $h\epsilon\Sigma_0{}^f$ the Fourier series $\sum \alpha_k g_k$ formed with $\alpha_k=(h, g_k)$ converges in norm to h, in other words, the Parseval equation $\sum|\alpha_k|^2 = \|h\|^2$ holds. Strong approximation (i.e., with respect to length) of vectors $h\epsilon\Sigma_0{}^f$ by finite sums $\sum_{k=1}^{n} \alpha_k' g_k$ necessitates a new axiom. Let $\xi(s)$ be any f-continuous numerical function on σ and g a vector such that sg is an f-continuous vector function of s (f-continuous, i.e., continuous in σ topologized by f). Then $g_\xi = M\{\xi(s)sg\}$ shall satisfy $|g_\xi|^2 \leqq M\{|\xi(s)|^2\}\|g\|^2$.

Special applications are: H. Weyl's approach to H. Bohr's theory of almost periodic functions [Math. Ann. 97, 338–356 (1926)]; the theories of F. Peter and H. Weyl [Math. Ann. 97, 737–755 (1927)], É. Cartan [Rend. Circ. Mat. Palermo 53, 217–252 (1929)], J. von Neumann [Trans. Amer. Math. 36, 445–492 (1934)], H. Weyl [Ann. of Math. (2) 35, 486–499 (1934)]. Also the theory of vector-valued almost periodic functions in a group with values from a Hilbert space H (without denumerability axiom) comes under the general theory. Such functions were introduced (with more general H) by S. Bochner and J. von Neumann [Trans. Amer. Math. Soc. 37, 21–50 (1935)].

In an appendix on the approach of Maak [Abh. Math. Sem. Hamburg. Univ. 11, 240–244 (1935); 367–380 (1936)] to the theory of almost periodic functions, the combinatorial lemma used by Maak is interpreted as a marriage problem; the proof given is a simplification of Maak's. *E. Følner.*

Bodiou, Georges. Renforcement des relations d'incertitude en statistique quantique par l'introduction d'un coefficient complexe de corrélation. C. R. Acad. Sci. Paris 228, 540–542 (1949).

Let f be a fixed vector of norm one in Hilbert space, and write, for any self-adjoint linear operator A, $E(A)=(Af, f)$, $\sigma^2(A)=E(A-E(A))^2=E(A^2)-E^2(A)$, and $\delta A = A-E(A)$, and, if B is also a self-adjoint linear operator, write $[A, B]=AB-BA$ and $[A, B]_+=AB+BA$. The author sharpens the known inequality $2\sigma(A)\sigma(B)\geqq|E([A, B])|$ by proving that $4\sigma^2(A)\sigma^2(B)\geqq E^2([\delta A, \delta B]_+)+|E^2([A, B])|$.

P. R. Halmos (Chicago, Ill.).

Rothe, E. H. Completely continuous scalars and variational methods. Ann. of Math. (2) 49, 265–278 (1948).

In an earlier paper [same Ann. (2) 47, 580–592 (1946); these Rev. 8, 158] the author defined the notions of the gradient of a scalar mapping in a real Hilbert space and of completely continuous scalar mappings. Among the results proved in the present paper are: (i) if the scalar mapping $I(x)$ is defined and completely continuous in the sphere V $(\|x\|\leqq r)$, then $I(x)$ assumes a maximum and a minimum

in V; (ii) if in addition $I(x)$ has the gradient $F(x)$, the equation $F(x)=\lambda x$ has two different solutions belonging to real values of λ; (iii) under the conditions of (i), $i(x)=\frac{1}{2}(x, x)+I(x)$ attains a minimum in V. Applications to integral equations and to Sturm-Liouville problems are indicated. *F. Smithies* (Cambridge, England).

LaSalle, J. P. Singular measurable sets and linear functionals. Math. Mag. **22**, 67–72 (1948).

The space $L^{\omega}(X, m)$ is the intersection of all $L^p(X, m)$, $p\geqq 1$, with the intersection topology. It is a convex complete metrizable topological linear algebra, with the obvious multiplication. Its interest lies in the fact that, when X is not singular, it has no continuous complex-valued linear homomorphisms, or what comes to the same thing, no proper subsets U open and convex for which UU is contained in U [R. Arens, Bull. Amer. Math. Soc. **52**, 931–935 (1946); these Rev. **8**, 165]. In the present paper, singular X are considered, and a correspondence between singular subsets of X and continuous complex-valued linear homomorphisms of $L^{\omega}(X, m)$ is shown. The method is also applied to the $L^p(X, m)$, $0<p<1$. *R. Arens.*

Cameron, R. H., and Hatfield, C., Jr. On the summability of certain orthogonal developments of nonlinear functionals. Bull. Amer. Math. Soc. **55**, 130–145 (1949).

It is known that the Fourier-Hermite functionals form a set of closed orthonormal functionals in the space of (linear or nonlinear) functionals $F[x]$ which are Wiener measurable and whose squares are Wiener summable. The general set of Fourier-Hermite functionals involves a complete orthonormal set $\{\alpha_j(t)\}$ on the interval $0\leqq t\leqq 1$. On forming the Fourier-Hermite functionals with the specialized set $\alpha_j(t)=2^{\frac{1}{2}}\cos(j-\frac{1}{2})\pi t$ the authors prove that the Fourier-Hermite series for a bounded Wiener measurable functional $F[x]$ (linear or nonlinear) possesses the property that it is summable by infinite-dimensional Abel means at any point x_0 at which the functional $F[x]$ is continuous in the Hilbert topology As a corollary they obtain a result on point-wise convergence of the series. The proof uses a classical formula for sums of products of Hermite functions, together with a careful analysis of the difference between a functional and its Fourier-Hermite partial sums. *W. T. Martin.*

Korenblyum, B. I. On certain special commutative normed rings. Doklady Akad. Nauk SSSR (N.S.) **64**, 281–284 (1949). (Russian)

The author presents results concerning the linear functionals on and the ideals in certain Banach algebras of functions on locally compact separable Abelian groups, multiplication being defined as convolution. From these results various Tauberian theorems, including the second of Wiener's general Tauberian theorems, are asserted to follow. If G is such a group, Ξ the class of nonnegative Baire functions in $L_1(G)$ (i.e., integrable relative to Haar measure), Ξ^* the subset of Ξ consisting of bounded functions, $M^p(G, \theta)$ the space of measurable complex-valued functions ψ on G for which the norm $\text{ess sup}_{y\epsilon G}(\int\theta(xy)|\psi(y)|^p dx)^{p-1}$ is finite, and R^p the second conjugate space of M^p, then R^p may be identified with a class of complex-valued functions on G. Then R^p is an ideal in $L_1(G)$, multiplication being convolution, and a Banach algebra relative to its own norm. If $\theta\epsilon\Xi^*$ and $p>1$, every maximal ideal other than R^p in the algebra R'^p obtained from R^p by adjoining an identity arises from a unique character of G, in the usual way, and an ideal in R^p is dense in R^p if and only if for each point of the character group of G there is an element of the ideal whose Fourier transform does not vanish at the point; there are analogous results for $p=1$.

$\cdot(\theta\epsilon\Xi)$

An explicit integral representation is given for the continuous linear functionals on R^p. This is used to extend Godement's modification [described inaccurately by the author as a generalization] of Beurling's theorem [Godement, C. R. Acad. Sci. Paris **223**, 16–18 (1946); these Rev. **8**, 14; Beurling, Acta Math. **77**, 127–136 (1945); these Rev. **7**, 61] to R^p for $p>1$ and $\theta\epsilon\Xi^*$. The results which concern the ideal theory of R^p, and also a class of Tauberian theorems which includes the cited theorem of Wiener's, follow from results contained in a paper by the reviewer [Trans. Amer. Math. Soc. **61**, 69–105 (1947); these Rev. **8**, 438, especially theorem 3.1, which is stated for the case of the additive group of the reals, but which is valid, with assumption (i) of that theorem, for an arbitrary locally compact Abelian group], which was apparently not known to the author. *I. E. Segal* (Chicago, Ill.).

Tortrat, Albert. Opérateurs linéaires bornés, dans un espace de Banach, pour un pôle de la résolvante. C. R. Acad. Sci. Paris **228**, 638–640 (1949).

Propositions are stated without proof concerning the decomposition of a Banach space E in terms of the null manifolds and ranges of successive powers of the operator $B=\lambda I-A$, where λ is a pole of order ν of the resolvent of A. It is assumed that the null manifold of B is of finite dimensions. *J. L. B. Cooper* (London).

Yosida, Kôsaku. On the differentiability and the representation of one-parameter semi-group of linear operators. J. Math. Soc. Japan **1**, 15–21 (1948).

For $0\leqq s<\infty$ let $\{U_s\}$ be a semi-group of linear operators on a complex Banach space E such that $U_0=I$, $\|U_s\|\leqq 1$ and $\lim_{s\to t} U_s x=U_t x$, $x\epsilon E$. The author investigates properties of the operator A given by $Ax=\text{weak}\lim_{h\to 0} h^{-1}(U_h-I)x$. Among other properties it is shown that A is closed on its domain of definition D which is a dense linear manifold and that for $x\epsilon D$, $\lim_{h\to 0} h^{-1}(U_{t+h}-U_t)x=AU_t x=U_t Ax$. He obtains a characterization for operators A satisfying these properties for some semi-group as above. [A note added in proof states that E. Hille [C. R. Acad. Sci. Paris **225**, 445–447 (1947); these Rev. **9**, 193] also obtained essentially the same results by a different method. Complete details of a characterization of the operators A under the conditions above are in Hille's book [Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloquium Publ., v. 31, New York, 1948; these Rev. **9**, 594], which appeared after the present paper was written.] The author uses his results to demonstrate Stone's theorem on the representation of groups of unitary operators on Hilbert space. *B. Yood.*

Rohlin, V. On dynamical systems whose irreducible components have a pure point spectrum. Doklady Akad. Nauk SSSR (N.S.) **64**, 167–169 (1949). (Russian)

The author announces some further results of his study of measure-preserving transformations. In this note heavy use is made of the terminology and results of his preceding notes [same Doklady (N.S.) **58**, 29–32, 189–191 (1947); these Rev. **9**, 230] and of the exposition of the Hellinger-Hahn multiplicity theory as presented by the author in

collaboration with Plesner [Uspehi Matem. Nauk (N.S.) 1, no. 1(11), 71–191 (1946); these Rev. 9, 43]. Denote by Σ the class of all automorphisms T with the property that (almost) all their irreducible (metrically transitive) parts have pure point spectrum. If C is the generic symbol for the elements of the canonical decomposition space of some T in Σ, and if, for any irreducible T with pure point spectrum, $G(T)$ is the group of proper values of T, then the function $G_C=G(T_C)$ uniquely determines the automorphism type of T. Call a not more than countably valued numerical function measurable if it may be decomposed into single valued measurable branches; in this language a necessary and sufficient condition that a given function G_C arise from some T is that it be measurable. The second half of the note is devoted to a rather complicated characterization of the spectral behavior of the unitary operators induced by some T in Σ. As an application of his characterization the author mentions that if the spectrum of an automorphism of class Σ contains an absolutely continuous component, then it contains the Hellinger type of Lebesgue measure with infinite multiplicity. No proofs are given.

P. R. Halmos (Chicago, Ill.).

Theory of Probability

Skellam, J. G. A probability distribution derived from the binomial distribution by regarding the probability of success as variable between the sets of trials. J. Roy. Statist. Soc. Ser. B. 10, 257–261 (1948).

Let the probability of success p vary from set to set as $\varphi(p)=p^{\alpha-1}(1-p)^{\beta-1}/B(\alpha,\beta)$, $0\leqq p\leqq 1$, but let p remain the same for all trials in the same set; then the probability of obtaining x successes in n trials is shown to be ${}_nC_xB(\alpha+x,\beta+n-x)/B(\alpha,\beta)$. The factorial moments, the estimation of the parameters α and β by the method of maximum likelihood, the relation of the distribution to the negative binomial, and some numerical examples of the fitting of the distribution are given. *L. A. Aroian.*

Mazurkiewicz, S. Un théorème sur les fonctions caractéristiques. Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1940–1946, 1–3 (1948).

Let X and Y be independent random variables and $Z=X+Y$. Denote the corresponding characteristic functions by $\phi_x(t)$, $\phi_y(t)$, and $\phi_z(t)=\phi_x(t)\phi_y(t)$. If $\phi_z(t)$ is analytic in the circle $|t|<r$ so is $\phi_x(t)$; if $\phi_z(t)$ is entire of order not exceeding ρ, so is $\phi_x(t)$. For the proof it may be assumed that 0 is a median value for y. It follows easily that $\Pr\{|Z|>\lambda\}\geqq\frac{1}{2}\Pr\{|X|>\lambda\}$ and hence $M_{2k}(Z)\geqq\frac{1}{2}M_{2k}(X)$, where M_{2k} denotes the moment of order $2k$. The assertion follows easily by comparing coefficients in the two Taylor series. *W. Feller* (Ithaca, N. Y.).

✱Pólya, G. Remarks on characteristic functions. Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 115–123. University of California Press, Berkeley and Los Angeles, 1949. \$7.50.

L'auteur démontre que: (a) toute fonction $f(t)$ de la variable rèelle t, définie, rèelle et continue dans $(-\infty,\infty)$, telle que: $f(0)=1$, $\lim_{t\to\infty}f(t)=0$, $f(-t)=f(t)$, et convexe pour $t>0$, est la caractéristique d'une fonction de répartition $F(x)$ absolument continue dont la dérivée est une fonction paire, et continue sauf peut être pour $x=0$; applications à la divisibilité des lois de probabilités; (b) si $G(x)$ est une fonction définie et à variation bornée sur $(-\infty,\infty)$, une condition nécessaire et suffisante pour que $G(x)$ soit constante sauf sur un intervalle fini est que sa transformée de Fourier $g(t)=\int_{-\infty}^{\infty}e^{itx}dG(x)$, considerée comme fonction de la variable complexe t, soit une fonction entière et de type exponentiel; en outre, si $(-k',k)$ est le plus petit intervalle hors duquel $G(x)$ est constant, on a:

$$k=\limsup_{r\to\infty} r^{-1}\log|g(-ir)|,\quad k'=\limsup_{r\to\infty} r^{-1}\log|g(ir)|;$$

applications aux cas où $G(x)$ est une fonction de répartition, ou la différence de 2 fonctions de répartition.

R. Fortet (Caen).

Fisher, Ronald Aylmer, et Dugué, Daniel. Un résultat assez inattendu d'arithmétique des lois de probabilité. C. R. Acad. Sci. Paris 227, 1205–1206 (1948).

Let X and Y be independent random variables with the characteristic function $\phi(z)=\{\frac{2}{5}\cos 2z+\frac{4}{5}\cos z-\frac{1}{5}\}e^{-z^2/2a^2}$, where the quantity $e^{-a^2/2}=u$ is a root of the equation $2u^4+4u=1$. The corresponding distribution function $F(x)$ is obviously a linear combination of five normal distributions with means ±2, ±1 and 0. One gets $F'(0)=0$ so that $F(x)$ does not contain a normal component. On the other hand, the form of $\phi^2(z)$ shows that $X+Y$ has the same distribution as $U+V$ where U and V are independent, U is normal with mean 0 and variance $2/a^2$ and $\Pr\{V=\pm4\}=\frac{1}{25}$, $\Pr\{V=\pm3\}=\frac{4}{25}$, $\Pr\{V=\pm2\}=\frac{2}{25}$, $\Pr\{V=0\}=\frac{11}{25}$.

W. Feller (Ithaca, N. Y.).

Segerdahl, C.-O. Some properties of the ruin function in the collective theory of risk. Skand. Aktuarietidskr. 31, 46–87 (1948).

The author continues his investigations of the collective theory of risk [Skand. Aktuarietidskr. 25, 43–83 (1942); these Rev. 8, 215; Uppsala thesis, 1939] by examining the effects of reinsurance. He assumes that a fixed sum is retained and the excess reinsured. Therefore the risk reserve is not debited beyond a certain amount which is independent of the sum due. The properties of the ruin function are studied, especially the behaviour of this function when reinsurance is gradually increased. The special risk sum distribution $s(z)=1-e^{-z}$ is discussed in detail in the second part of the paper. *E. Lukacs* (China Lake, Calif.).

Ammeter, H. A generalization of the collective theory of risk in regard to fluctuating basic-probabilities. Skand. Aktuarietidskr. 31, 171–198 (1948).

This paper considers the risk business of an insurance company, the premiums payable to the company and the risk sums paid by it. It is assumed that the expected number K of claims during a unit period is PQ, where P is a constant and Q is a chance variable with zero density for negative q, and density $h_0^{h_0}e^{-h_0q}q^{h_0-1}/\Gamma(h_0)$ for positive q. Here h_0 is a parameter of the distribution, which measures the "precision." It appears to the reviewer that the number of claims in a period is assumed to have the Poisson distribution. The claims in different unit periods are assumed independent. The author studies the amount to be paid out in a unit period and in n such periods, and considers various special cases and limiting cases. *J. Wolfowitz.*

Saxén, Tryggwe. On the probability of ruin in the collective risk theory for insurance enterprises with only negative risk sums. Skand. Aktuarietidskr. 31, 199–228 (1948).

The author studies insurance companies with negative risk sums (i.e., the sums are paid to the company, as in the case of annuities), whose distribution function is $s(z)$ assumed independent of the time t. The loading factor λ is assumed constant. The probability of a claim in a time interval of length Δt is $p\Delta t$, with p a constant, and the probability of the occurrence of more than one claim in the interval is $o(\Delta t)$. The author proves that the probability $\psi(u)$ of eventual ruin of the company, when the risk reserve is initially u, is $\exp(Cu)$, where C is the unique negative root of the equation

$$C(1-\lambda)+1=\int_{-\infty}^{0} \exp(-Cz)ds(z)$$

[this result is due to C.-O. Segerdahl, Skand. Aktuarietidskr. 25, 43–83 (1942), in particular, p. 74; these Rev. 8, 215]. Let $\psi(u, Q)$ be the probability of ruin at or before a time when a net premium Q has flowed through the company. The author proves that $\psi(u, Q)=\sum_{k=0}^{\infty}\varphi_k(u, 0, Q)$, where

$$\varphi_0(\xi, \tau, Q)=\epsilon(\xi, (1-\lambda)Q)e^{-(\xi/(1-\lambda)-\tau)},$$

$$\varphi_k(\xi, \tau, Q)=\int_{\tau}^{\xi/(1-\lambda)} e^{-(t-\tau)}\left\{\int_{-[(1-\lambda)Q-\xi]}^{0}\varphi_{k-1}(\xi-y, t, Q)ds(y)\right\}dt$$

and $\epsilon(\xi, x)=1$ for $x\geqq\xi$, $\epsilon(\xi, x)=0$ for $x<\xi$. This result is applied to the special case $s(y)=\epsilon(-1, y)$. Finally let $G(u, h)=\int_0^{\infty}e^{hQ}d_Q\psi(u, Q)$. The author obtains $\log G(u, h)$ as the sum of an infinite series. Essentially this result was obtained by Segerdahl [Uppsala thesis, 1939].

J. Wolfowitz (New York, N. Y.).

Anderson, T. W., and Rubin, Herman. Estimation of the parameters of a single equation in a complete system of stochastic equations. Ann. Math. Statistics 20, 46–63 (1949).

Given a system of stochastic difference equations

$$B_{yy}y_t'+\Gamma_{yz}z_t'=\epsilon_t', \qquad t=1, \cdots, T,$$

where y_t, z_t, ϵ_t (the transposes of y_t', z_t', ϵ_t') are random (row) vectors; y_t is observed; the coordinates of z_t are either coordinates of $y_{t-1}, y_{t-2}, \cdots$ or given constants; $\epsilon_1, \cdots, \epsilon_T$ are independent normal with zero means; B_{yy} and Γ_{yz} are constant matrices, the former square and nonsingular. The paper deals with the estimation of the coefficients of a single equation $\beta_y y_t'+\gamma_z z_t'=\zeta_t$ of the system under the assumption that $\beta_y=(\beta, 0)$, $\gamma_z=(\gamma, 0)$, where the number H of nonzero components of β_y and the number D of zero components of γ_z satisfy $D\geqq H-1$. By a method used in discriminant function theory the authors derive maximum likelihood estimates of β, γ; tests of whether D components of γ_z are zero; confidence regions for β, γ (under certain restrictions). A procedure for computing the maximum likelihood estimates is outlined.

W. Hoeffding.

***Arley, Niels. On the Theory of Stochastic Processes and Their Application to the Theory of Cosmic Radiation.** John Wiley & Sons, Inc., New York, N. Y., 1948. 240 pp. $5.00.

This is a reprint of the author's Copenhagen thesis [1943]; cf. these Rev. 7, 209.

Wold, H. On stationary point processes and Markov chains. Skand. Aktuarietidskr. 31, 229–240 (1948).

Let $\cdots, x_0, x_1, \cdots$ be nonnegative random variables defining a stationary process. The x_j's are to be considered as the times between successive events. It is supposed that the x_j's form a simple or multiple Markov process, so that the classical theorems on the asymptotic character of Markov transition probabilities (applicable to multiple Markov processes since these are simple processes with multi-dimensional state spaces) can be applied.

J. L. Doob (Urbana, Ill.).

Shannon, Claude E. Communication in the presence of noise. Proc. I.R.E. 37, 10–21 (1949).

Some of the considerations of another paper by the author [Bell System Tech. J. 27, 379–423, 623–656 (1948); these Rev. 10, 133] are discussed from a more geometric point of view.

J. L. Doob (Urbana, Ill.).

Burgers, J. M. Spectral analysis of an irregular function. Nederl. Akad. Wetensch., Proc. 51, 1222–1231 (1948). (English. Esperanto summary)

The author continues the work of his previous paper [same Proc. 51, 1073–1076 (1948); these Rev. 10, 311], studying now the fluctuations of a galvanometer driven by a signal derived from a random noise after filtering and squaring. (The squaring is due to a thermocouple.) The deflection z depends on time and defines a stationary stochastic process whose covariance function the author calculates, making various approximations. The author's hypotheses are obscure. If the original noise is Gaussian, as is usually supposed, even exact calculations become trivial, due to the fact that the covariance function of $x(t)^2$ is twice that of the square of $x(t)$, if the $x(t)$ process is Gaussian. [Cf. S. O. Rice, Bell System Tech. J. 24, 46–156 (1948), especially p. 89; these Rev. 6, 233.]

J. L. Doob (Urbana, Ill.).

Wolfowitz, J. The distribution of plane angles of contact. Quart. Appl. Math. 7, 117–120 (1949).

Suppose that particles A and B move with constant velocity in a plane. The point A moves with a closed convex curve to which it is rigidly attached, and is interior to it. The curve does not contain B initially. If during the motion the curve ever meets B the meeting can be characterized by the angles between the line AB and the directions of the two motions. The joint distribution of these two angles is found, on the hypothesis that the particle B is initially in any subregion g of a sufficiently large region G with probability proportional to the area of g.

J. L. Doob.

Chandrasekhar, S. On a class of probability distributions. Proc. Cambridge Philos. Soc. 45, 219–224 (1949).

Let a sequence of points be distributed in a Poisson spatial distribution, and let $r_1, r_2, \cdots$ be the vectors from a fixed origin to these points. The author calculates the distribution of the vector $F=\sum_j r_j/|r_j|^{n+1}$. The series converges if $n>3/2$. The limit distribution is spherically symmetric and stable, with exponent $3/n$. The asymptotic nature of the distribution of $|F|$ for large $|F|$ is found and the moments of $|F|$ of order less than $3/n$ are tabulated. The distribution of F arises in various physical and astronomical contexts [see, for example, Chandrasekhar, Astrophys. J. 94, 511–524 (1941); these Rev. 3, 281].

J. L. Doob.

Nuyens, Maurice, et Grosjean, Carl. Sur la diffusion des neutrons thermiques. C. R. Acad. Sci. Paris **228**, 245–246 (1949).

The stationary distribution of thermal neutrons in an infinite homogeneous medium is considered by the methods of the theory of random flights. If the trajectory of a particle consists of a sequence of n straight lines of lengths $l_1, \cdots, l_n$ randomly oriented, then it is known that the probability that the particle will be between r and $r+dr$ is given by

$$dP_n = 2\pi^{-1} r\,dr \int^{\infty} u^{-n+1} \sin ru \prod_{j=1}^{n} l_j^{-1} \sin l_j u\, du.$$

With $\lambda^{-1} e^{-l/\lambda} dl$ governing the probability distribution of the l's, the foregoing expression gives

$$\phi_n(r) = 2\pi^{-1} r \int_0^{\infty} \lambda^{-n} u^{-n+1} \sin ru \left(\int_0^{\infty} l^{-1} e^{-l/\lambda} \sin lu\, dl \right)^n du \qquad (n \geqq 2),$$

for the probability of occurrence, between r and $r+dr$, of a particle which has suffered exactly n collisions. Thus,

$$\phi_n(r) = 2\pi^{-1} \lambda^{-n} r \int_0^{\infty} (\tan^{-1} \lambda u)^n u^{1-n} \sin ru\, du.$$

The corresponding expression for the density of all particles which have suffered two or more collisions is given by

$$\rho(r) \approx \frac{1}{4\pi r^2} \sum_{n=2}^{\infty} \phi_n(r) = \frac{1}{2\pi^2 r} \int_0^{\infty} \frac{(\tan^{-1} \lambda u)^2}{\lambda u - \tan^{-1} \lambda u} \sin ru\, du.$$

This agrees with known results. *S. Chandrasekhar.*

Vajda, S. Introduction to a mathematical theory of the graded stationary population. Mitt. Verein. Schweiz. Versich.-Math. **48**, 251–273 (1948).

This paper is of an expository nature and concerns a constant population divided into k strata or grades with mortality and "forces of promotion" from a stratum to the next higher. [Cf. Biometrika **34**, 243–254 (1947); these Rev. **9**, 362]. *W. Feller* (Ithaca, N. Y.).

Clark, C. E. The statistical theory of the dead time losses of a counter. Rev. Sci. Instruments **20**, 51–52 (1949).

If in a counter μ impulses occur of which ν are recorded then μ is approximately normally distributed; the approximate variance of μ as a function of ν is given.

W. Feller (Ithaca, N. Y.).

Mathematical Statistics

Anscombe, F. J. The transformation of Poisson, binomial and negative-binomial data. Biometrika **35**, 246–254 (1948).

If r is a random variable with a Poisson distribution having a mean m, it is shown that for large values of m the value of c for which $(r+c)^{\frac{1}{2}}$ has a "most nearly" constant variance is $\frac{3}{8}$, i.e., a variance which is a constant except for terms of order m. The variance of $(r+\frac{3}{8})^{\frac{1}{2}}$ is $\frac{1}{4}+O(m^{-2})$. Similarly, if r is a random variable having a binomial distribution with mean m and total number of trials equal to n, it is shown that the values of c, e, and d_2 for which $(n+d_2)^{\frac{1}{2}} \sin^{-1} \{(r+c)/(n+d_1)\}^{\frac{1}{2}}$ has a "most nearly" constant variance for large n are $c=\frac{3}{8}$, $d_1=\frac{3}{4}$, $d_2=\frac{1}{2}$. The variance in this case is $\frac{1}{4}+O(n^{-2})$. Finally, it is shown that if r has the distribution

$$p(r) = \frac{\Gamma(r+k)}{r!\,\Gamma(k)} \left(\frac{m}{m+k} \right)^r (1+m/k)^{-k},$$

$r=0, 1, 2, \cdots$, then for large m and constant ratio k/m, $(k-\frac{1}{2})^{\frac{1}{2}} \sinh^{-1} \{(r+\frac{3}{8})/(k-\frac{3}{4})\}^{\frac{1}{2}}$ has variance $\frac{1}{4}+O(m^{-2})$.

S. S. Wilks (Princeton, N. J.).

Cramer, G. F. An approximation to the binomial summation. Ann. Math. Statistics **19**, 592–594 (1948).

The author is concerned with the approximate evaluation of the binomial sum $\sum_{i=r}^{s} \binom{n}{i} p^i q^{n-i}$ with the help of the Poisson sum $e^{-m} \sum_{i=r}^{s} m^i/i!$ for which, at present, more extensive tables are available. Instead of the usual choice $m=np$ the author suggests $m'=(n-r)p/(1-p)$, which depends on r, the index of the first frequency, as well as on n and p, so that different Poisson distributions would be used for the evaluation of the same binomial distribution at different variate values r. Gauges of accuracy are obtained from derived properties of the comparative rate of decrease of the tail area frequencies of the two distributions and illustrated with an example. *H. O. Hartley.*

David, F. N. Correlations between χ^2 cells. Biometrika **35**, 418–422 (1948).

The author discusses the correlations that may exist between the deviations of observed from calculated frequencies in the different cells of a table for the calculation of a χ^2. The result in the case of a single linear constraint, the preservation of totals, has long been known. The author shows how they may be calculated in the presence of additional linear conditions, if it can be assumed that the set of deviations obey a multivariate normal frequency law. It appears that these correlations, particularly for neighboring cells, are not always small; in fact for some quite ordinary cases, as the number of linear constraints is increased, they may tend toward 1 or -1 rather rapidly. *C. C. Craig.*

Lev, Joseph. The point biserial coefficient of correlation. Ann. Math. Statistics **20**, 125–126 (1949).

Let r be the sample point biserial correlation coefficient between a continuous variable y and a variable x which takes values 0 and 1, and let ρ denote the corresponding population parameter. If n represents the number of sample values of y, the author shows that $t=r(n-2)^{\frac{1}{2}}(1-r^2)^{-\frac{1}{2}}$ has a noncentral t distribution, and uses this result to test the hypothesis $\rho=0$, and to find confidence limits for ρ.

L. A. Aroian (New York, N. Y.).

Geary, R. C. Determination of linear relations between systematic parts of variables with errors of observation the variances of which are unknown. Econometrica **17**, 30–58 (1949).

Letting $x_{it}=x'_{it}+x''_{it}$ $(i=1, 2, 3; t=1, \cdots, n)$ be random variables ruled by the usual assumptions of confluence analysis, suppose $x'_{1t}=\alpha x'_{2t}$, while x_{3t} are regarded as instrumental in the sense of O. Reiersöl [Ark. Mat. Astr. Fys. **32A**, no. 4 (1945); these Rev. **7**, 317], so that

$$a = \text{cov}\,(x_{1t}, x_{3t})/\text{cov}\,(x_{2t}, x_{3t})$$

is an estimate of α. The author's main result is that a certain elementary transformation carries a into a variable which is distributed as Gosset and Fisher's t with $n-1$ degrees of freedom. Assuming that x'_{1t}, x'_{2t} and x_{3t} are non-

$x'_{1t} = \alpha x'_{2t}$

random (model B), the same result is shown to be approximately valid for large n. There are remarks on the case of several variables and a numerical illustration. [The author refers to model B as appropriate for economic time series; this is somewhat obscure to the reviewer, for it seems that the autocorrelation which bedevils the significance problems in the analysis of such series should be taken into account even in the disturbances x''_{ii}.] *H. Wold* (Uppsala).

Godwin, H. J. A note on Kac's derivation of the distribution of the mean deviation. Ann. Math. Statistics **20**, 127 (1949).

The author has found a formula for the density function of the mean deviation in samples from a normal population [Biometrika **33**, 254–256 (1945); these Rev. **8**, 42]. Subsequently the reviewer gave another derivation which resulted in a somewhat different formula [same Ann. **19**, 257–261 (1948); these Rev. **9**, 601]. The reviewer was unable to establish the equivalence of the two formulas and in the present note Godwin proves the equivalence. The proof is based on a result of a recent paper by Godwin [Biometrika **35**, 304–309 (1948); these Rev. **10**, 387]. *M. Kac.*

Cox, D. R. A note on the asymptotic distribution of range. Biometrika **35**, 310–315 (1948).

The author derives the asymptotic expression

$$c_n[\Phi(\tfrac{1}{2}w)-\Phi(-\tfrac{1}{2}w)]^{n-\frac{3}{2}}[\Phi'(\tfrac{1}{2}w)]^2[\Phi''(-\tfrac{1}{2}w)]^{-\frac{1}{2}}$$

for the frequency function of the range of a sample of n individuals drawn from a symmetrical and unimodular population with mean zero and distribution function $\Phi(x)$. In the normal case numerical comparison of means, standard deviations, and Pearson betas shows the approximation to be better than the Gumbel approximation [Ann. Math. Statistics **18**, 384–412 (1947); these Rev. **9**, 195] but less accurate than the reviewer's [Biometrika **34**, 111–119 (1946); these Rev. **8**, 395], the latter suffering, however, from the disadvantage of involving a nonlinear transformation of the range. *G. Elfving* (Helsingfors).

Kimball, Bradford F. An approximation to the sampling variance of an estimated maximum value of given frequency based on fit of doubly exponential distribution of maximum values. Ann. Math. Statistics **20**, 110–113 (1949).

A sample is taken from a distribution with cumulative distribution function $F(x)=\exp(-e^{-y})$, $y=\alpha(x-u)$. For fixed y_0, let $g(u,\alpha)=u+y_0/\alpha$. The author derives an approximate variance for the maximum likelihood estimate of $g(u,\alpha)$, and uses this to obtain confidence intervals for $g(u,\alpha)$, which are shorter than the ones he gave in a previous paper [same Ann. **17**, 299–309 (1946); these Rev. **8**, 475]. *E. L. Lehmann* (Berkeley, Calif.).

Mood, A. M. Tests of independence in contingency tables as unconditional tests. Ann. Math. Statistics **20**, 114–116 (1949).

The author comments on the equivalence of conditional tests and tests similar to the sample space as constructed with the aid of sufficient statistics by Neyman [Philos. Trans. Roy. Soc. London. Ser. A. **236**, 333–380 (1937)]. *E. L. Lehmann* (Berkeley, Calif.).

Huzurbazar, V. S. Inverse probability and sufficient statistics. Proc. Cambridge Philos. Soc. **45**, 225–229 (1949).

The author's summary is as follows. It is shown that when the probability distribution of a variable x, depending on a parameter α, admits a sufficient statistic for α, the form of the posterior probability density function of α bears a close resemblance to the form of the probability density function of x itself. This resemblance holds good even when the range depends upon the parameter or when the distribution contains several parameters for which there are jointly sufficient statistics. *J. Wolfowitz.*

Radhakrishna Rao, C. Sufficient statistics and minimum variance estimates. Proc. Cambridge Philos. Soc. **45**, 213–218 (1949).

It is known [B. O. Koopman, Trans. Amer. Math. Soc. **39**, 399–409 (1936)] that, if a frequency function which depends on a parameter θ admits a sufficient statistic and satisfies certain regularity conditions, then the sample frequency function $\phi(x_1,\cdots,x_n;\theta)$ is of the form (1) $\phi=\exp(\Theta_1X_1+\Theta_2+X_2)$ wherever $\phi>0$, where Θ_1, Θ_2 are functions of θ only and X_1, X_2 are functions of $x_1,\cdots,x_n$ only. The author shows that if (a) a sufficient statistic exists and ϕ is of the form (1) and (b) $\int_{R_n}\phi dx_1\cdots dx_n=1$ can be differentiated twice with respect to Θ_1 under the integral sign, then X_1 is the unbiased minimum variance estimate of $-d\Theta_2/d\Theta_1$. Since X_1 is a sufficient statistic, a minimum variance unbiased estimate of a function of θ must be a function of X_1 only [C. Radhakrishna Rao, same Proc. **43**, 280–283 (1947); these Rev. **8**, 478]. It is shown that a function $F(X_1)$ is the only unbiased estimate of its expectation among the class of unbiased estimates of $EF(X_1)$ of the form $F(X_1)+f(X_1)$, where $f(x)$ is continuous and $Ef(X_1)$ can be differentiated any number of times with respect to Θ_1 under the integral sign. Sufficient conditions under which this holds are given. It follows that $F(X_1)$ is the minimum variance estimate under the above restriction on the class of estimates. This theorem applies to cases where the variance of $F(X_1)$ exceeds the lower bound given by an inequality of Cramér and the author [H. Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, p. 480; these Rev. **8**, 39]. *W. Hoeffding.*

Wald, A., and Wolfowitz, J. Bayes solutions of sequential decision problems. Proc. Nat. Acad. Sci. U. S. A. **35**, 99–102 (1949).

Let $r(\xi,D)$ denote the risk from using a sequential decision process D when the unknown parameter has a priori distribution ξ. A process D is called a Bayes solution for ξ if it minimizes $R(\xi,D)$ for the given ξ. A necessary and sufficient condition for D to be a Bayes solution is given for the case of independent observations at constant cost. The condition is that D should require (1) continued sampling as long as, for the a posteriori distribution ξ^*, there is a sequential continuation lowering the risk below its present level, (2) stopping when, for ξ^*, the risk of any sequential continuation is above its present level, and that, when sampling stops, the decision should be the best possible for the resulting ξ^*. The class of ξ^*'s of type (2) with a given decision is convex. Proofs and applications will be given elsewhere. *D. Blackwell* (Washington, D. C.).

Baker, G. A. The variance of the proportions of samples falling within a fixed interval for a normal population. Ann. Math. Statistics **20**, 123–124 (1949).

The author compares the relative efficiency of two different estimates of the proportion of a normal distribution contained between two fixed limits L_1 and L_2. The first estimate is the proportion in the observed sample falling

between these limits; the second estimate is given by $s^{-1}(2\pi)^{-\frac{1}{2}}\int_{L_1}^{L_2} e^{-\frac{1}{2}(x-\bar{x})^2/s^2}dx$, where $\bar{x}$ and s are the mean and standard deviation of the sample. Some numerical results indicate the greater efficiency of the second estimate.

E. Paulson (Seattle, Wash.).

***Barankin, Edward W. Extension of the Romanovsky-Bartlett-Scheffé test.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 433–449. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

Let $\xi_1, \cdots, \xi_m$; $\eta_1, \cdots, \eta_n$ $(m \leqq n)$ be known numbers, and $x_1, \cdots, x_m$; $y_1, \cdots, y_n$ a sample of independent random variables, the distribution of x_i and y_j being normal, $(h_1+k_1\xi_i, \sigma_1)$ and $(h_2+k_2\eta_j, \sigma_2)$, respectively; here $h_1, k_1, \sigma_1, h_2, k_2, \sigma_2$ are unknown parameters. It is required to find an exact, unbiased test for the hypothesis $k_1=k_2$. The answer given constitutes a generalization of Scheffé's solution of the Fisher-Behrens problem [Ann. Math. Statistics **14**, 35–44 (1943); these Rev. **4**, 221], the criterion being t-distributed with $m-2$ degrees of freedom. Vector methods are used.

G. Elfving (Helsingfors).

Peiser, A. M. Correction to "Asymptotic formulas for significance levels of certain distributions." Ann. Math. Statistics **20**, 128–129 (1949).

The paper appeared in the same Ann. **14**, 56–62 (1943); these Rev. **4**, 222.

Olds, Edwin G. The 5% significance levels for sums of squares of rank differences and a correction. Ann. Math. Statistics **20**, 117–118 (1949).

The author computes a table of pairs of values between which $\sum d^2$ has a probability .95 of being included under the hypothesis that the respective rankings are unrelated, where $\sum d^2$ represents the sum of squares of the rank differences. This is an addition to a table previously published by the author [same Ann. **9**, 133–148 (1938)]. An error in this previous paper is corrected.

H. Chernoff.

Paulson, Edward. A multiple decision procedure for certain problems in the analysis of variance. Ann. Math. Statistics **20**, 95–98 (1949).

Let $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k$ be varietal means in increasing order, each based on r observations, from the same experiment, which have error variances σ^2 estimated by s^2. The varieties may be classified into two groups by the criterion: put $\bar{x}_i$ in the superior group if $\bar{x}_i > \bar{x}_k - \lambda s/\sqrt{r}$, where λ is a constant determined so that the probability of not including all varieties in the superior group has a specified (small) value when all varieties are the same; λ is readily found with the aid of tables of the Studentized range. An expression is given for the power of the criterion in the case all varieties but one are the same.

A. M. Mood.

Kishen, K. On fractional replication of the general symmetrical factorial design. J. Indian Soc. Agric. Statistics **1**, 91–106 (1948).

The author establishes explicitly the correspondence between group theoretical methods and the methods of finite geometries used in constructing confounded symmetrical factorial designs. He establishes a correspondence between the $(m-k-1)$-flats at infinity of the PG$(m\cdot s)$ and subgroups H of order s^k of the effect group. The treatment subgroup of order s^{m-k} of treatments orthogonal to H is then given by the s^{m-k} treatment combinations corresponding to the finite points lying on the $(m-k)$-flat which passes through $(0, \cdots, 0)$ and the $(m-k-1)$-flat corresponding to H. If an effect group of order s^k is taken as the alias subgroup a $1/s^k$ fractional replicate may thus be constructed. Confounded fractional replications may also be obtained in this way. The author illustrates his method by constructing a $\frac{1}{16}$ replicate of a 4^4 design and a $\frac{1}{4}$ replicate of a 4^5 design.

H. B. Mann (Columbus, Ohio).

TOPOLOGY

Colmez, Jean. Sur les espaces à écarts. C. R. Acad. Sci. Paris **228**, 156–158 (1949).

There are presented several generalizations of the notion of metric, some having the same content as the uniform structure of Weil, but presumably conceptually different.

R. Arens (Los Angeles, Calif.).

Novák, Jos. Construction d'espaces dont les points O- resp. $\bar{\bar{O}}$-séparables sont donnés d'avance. Časopis Pěst. Mat. Fys. **73**, 49–57 (1948). (Czech. French summary)

The author calls two points of a topological space O-separated ($\bar{\bar{O}}$-separated) if they possess disjoint neighborhoods (closed neighborhoods). If M is a T_1-space [cf. Alexandroff and Hopf, Topologie, vol. 1, Springer, Berlin, 1935] let $\varphi(x)$, for any $x \epsilon M$, consist of all $y \epsilon M$ such that x, y are not O-separated. A sufficient condition is given for the existence of a T_1-space (on a given set M) with prescribed sets $\varphi(x)$. In the second part of the paper, a construction is given, for any $\mathfrak{m} > \aleph_0$, of a Hausdorff space of power $\mathfrak{m}$ containing no $\bar{\bar{O}}$-separated points. This solves a problem of Alexandroff and Urysohn [Verh. Akad. Wetensch Amsterdam, Afd. Natuurk. Sect. 1. **14**, no. 1 (1929)].

M. Katětov (Prague).

Novák, Josef. Regular space, on which every continuous function is constant. Časopis Pěst. Mat. Fys. **73**, 58–68 (1948). (Czech. English summary)

A problem of Urysohn [Math. Ann. **94**, 262–295 (1925)] is solved by constructing, for an arbitrary $\mathfrak{m} > \aleph_0$, a regular topological space of power $\mathfrak{m}$ in which every continuous real function is constant. The problem has already been solved (for every $\mathfrak{m}$ which is not a sum of $\aleph_0$ cardinal numbers less than $\mathfrak{m}$) by E. Hewitt [Ann. of Math. (2) **47**, 503–509 (1946); these Rev **8**, 165]. The author's solution was found independently, and his method is different, making use of L-spaces [cf. Fréchet, Les Espaces Abstraits, Gauthier-Villars, Paris, 1928].

M. Katětov (Prague).

Dolcher, Mario. Due teoremi sull'esistenza di punti uniti nelle trasformazioni piane continue. Rend. Sem. Mat. Univ. Padova **17**, 97–101 (1948).

Let $\bar{C}$ be a disk in E^2 bounded by the circle C^*. Let Φ be a continuous mapping $\bar{C} \rightarrow E^2$. It is proved that if $\Phi(C^*)$ fails to meet $\bar{C}$ and if the index of $\Phi(C^*)$ relative to points of C^* is different from zero, Φ admits a fixed point. This is a simple consequence of known theorems.

P. A. Smith.

Finzi, Arrigo. Sur le problème de la génération d'une transformation donnée d'une courbe fermée par une transformation infinitésimale. C. R. Acad. Sci. Paris 228, 531–533 (1949).

Finzi, Arrigo. Un théorème sur les familles de transformations régulières. C. R. Acad. Sci. Paris 228, 631–633 (1949).

In the first paper the author investigates the conditions under which a homeomorphism $T:x'=g(x)$ of a simple closed curve onto itself can be obtained as a member of a one-parameter continuous group, having differential equation $dx/dt=\xi(x)$. It is remarked that a necessary condition is that no power of T, other than one which yields the identity, have fixed points. It is further stated that, if a power of T is the identity, then T can be obtained from infinitely many groups, provided that $g(x)$ has a continuous derivative. The transformation T determines a rotation number k. It is stated that, if k is irrational, then there is at most one group containing T, and such a group does exist if $g(x)$ has a second derivative satisfying a Lipschitz condition and the continued fraction expansion of k satisfies a certain condition. Finally, it is stated that, for a one-parameter family $x'=g(x,\theta)$ of such transformations, the measures of the sets of values of θ for which k is rational or for which k is irrational and a group exists are positive.

In the second paper, the family $T(\theta):x'=g(x,\theta)$ is again considered, with rotation numbers $k(\theta)$. It is asserted that, if the difference quotient of $k(\theta)$ is bounded away from zero and the partial derivatives $\partial g/\partial x$, $\partial^2 g/\partial x^2$, $\partial g/\partial\theta$, $\partial^2 g/\partial x\partial\theta$ exist and satisfy Lipschitz conditions, then there is a family of groups $G(\theta)$, with differential equations $dx/dt=\xi(x,\theta)$, such that $G(\theta)$ contains $T(\theta)$. The proof, which is sketched, uses the continued fraction expansion of $k(\theta)$.

W. Kaplan (Zurich).

Wang, Hsien-Chung. The homology groups of the fibre bundles over a sphere. Duke Math. J. 16, 33–38 (1949).

Let X be a fibre bundle over a sphere S^n. Let the fibre F (which is a connected polyhedron) have $(n-1)$th homology group $H^{n-1}(F)$ isomorphic to the integers, and higher ones vanishing. Then $H^r(X)\approx H^r(S\times F)$, except possibly for $r=n-1$ and n; if not, then $H^{n-1}(X)\approx$ integers mod m for some m, and $H^n(X)\approx 0$. Two similar theorems are given. The results overlap with those of A. Lichnerowicz [C. R. Acad. Sci. Paris 227, 711–712 (1948); these Rev. 10, 203].

H. Whitney (Cambridge, Mass.).

Jones, F. Burton. A note on homogeneous plane continua. Bull. Amer. Math. Soc. 55, 113–114 (1949).

There exists a homogeneous compact plane continuum which is not a simple closed curve [R. H. Bing, Duke Math. J. 15, 729–742 (1948); these Rev. 10, 261]. This contradicts the statement by G. Choquet [C. R. Acad. Sci. Paris 219, 542–544 (1944); these Rev. 7, 335] that a homogeneous compact plane continuum M must be a simple closed curve. The author shows that if in addition it is assumed either (1) that M is aposyndetic or (2) that M contains no weak cut point, then M is a simple closed curve. Definitions: a continuum M is aposyndetic at a point x in M if for each y in $M-x$ there exists a subcontinuum H of M and an open set U of M such that $M-y\supset H\supset U\supset x$; if it is aposyndetic at each of its points, then M is aposyndetic; a point x of M is a weak cut point of M if $M-p$ is not continuum-wise connected.

J. H. Roberts.

Horn, Alfred, and Valentine, F. A. Some properties of L-sets in the plane. Duke Math. J. 16, 131–140 (1949).

A set S, in the plane, is an L_n set if each pair of points of S can be joined by a polygonal line in S having at most n segments. Typical theorems are as follows. (1) If S is a bounded closed L_2 set, then each bounded component of the complement of S is an L_2 set, and the unbounded component of the complement of S is an L_4 set. (2) If S is a simply connected, bounded closed L_2 set, then S is a sum of convex sets every two of which have a point in common.

E. E. Moise (Ann Arbor, Mich.).

Wallace, A. D. Endelements and the inversion of continua. Duke Math. J. 16, 141–144 (1949).

This note considers two compact (=bicompact) connected Hausdorff spaces X and Y and a single-valued continuous transformation $f(X)=Y$. The introduction states "it will be shown that if f is non-alternating then there exists a proper subcontinuum of Y with a connected inverse," but the ~~proof~~ thelo given in the paper covers only the special case of this result when Y contains a cutpoint. The paper defines endelements and prime-chains for the spaces under consideration, and proves, among other things, that if X contains a cutpoint it contains an endelement.

D. W. Hall (College Park, Md.).

GEOMETRY

Schanzer, Roberto. Di un nuovo ordine logico nella geometria. (On a new logical order in geometry). Sigma, no. 8-9, 497–516 (1948).

Nach Ansicht des Verfassers sind Punkte, Gerade, Raum nicht die Begriffe, die sich dem Menschen ursprünglich darboten, sondern sie sind bereits Resultat eines wahrscheinlich unbewussten Prozesses während der vorwissenschaftlichen Periode des Denkens, der an anderen Begriffen seinen Ausgang nahm. Die vorliegende kurze Übersicht über eine vierteilige Arbeit des Verf. kündigt für den vierten Teil erst die strenge Definition von Gerade, Ebene und Raum und eine deduktive Herleitung ihrer Haupteigenschaften an aufgrund des im dritten Teil gewonnenen und hier nicht näher ausgeführten Begriffes der "subsolid objects." Der erste und zweite Teil soll allgemeine logische Principien und methodologische Untersuchungen behandeln. Der Inhalt des ersten Teils wird in einem Anhang der vorliegenden Arbeit näher ausgeführt.

R. Moufang.

Hudson, Douglas Rennie. Density and packing in an aggregate of mixed spheres. J. Appl. Phys. 20, 154–162 (1949).

When equal spheres of radius r are arranged so as to have coordination number $K=12$ the sphere centers are at the nodes of a solid tessellation of tetrahedra and octahedra [see W. W. R. Ball, Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, pp. 146–151]. The interstices between the spheres are therefore of two kinds, here called "triangular" and "square" holes. "With every sphere, one square and two triangular holes are associated." The purpose of the paper is to investigate

the increase in density when n equal spheres of radius R ($R<r$) are packed into these interstices. The equation used for calculating R for each value of n is given and tabulated along with the values of R/r and of the density increment $n(R/r)^3$, etc. The relation between the radius ratio and the increment in density is shewn graphically and the significance of the various peak densities is discussed. The space associated with each sphere in the original close packing (i.e., with $K=12$ and radius r) is erroneously given as $8r^3/3$. In the introductory summary of the paper it is remarked that the interstices "are connected by a continuous labyrinth through which a ball not exceeding $(2/\sqrt{3}-1)r$ in radius can be threaded." This expression for the coefficient of r is correct, but throughout the paper it is given the erroneous value 0.22475. *S. Melmore* (York).

Supnick, Fred. On the dense packing of spheres. Trans. Amer. Math. Soc. 65, 14–26 (1949).

Let $S_1,\cdots,S_n$ be a set of spheres with radii $r_1\leqq r_2\leqq\cdots\leqq r_n$, respectively. Let S be a right circular cylinder with radius r, such that $r_n\leqq r$. The spheres are said to be packed into S and form a packing when they are assigned positions wholly inside S such that no two have interior points in common. All packings considered in this paper are subject to the restriction that, if any three spheres S_i, S_j, S_k of the set of spheres be packed into S so that S_i is externally tangent to S_j, and S_j to S_k, then for all possible relative positions of the three spheres, S_i and S_k have no interior point in common. The density of the packing is defined as the volume ratio of the spheres to the smallest right circular frustrum of S and its interior which encloses the packing. A packing is said to be incompressible if any two adjacent spheres are tangent to each other and each tangent to S at diametrically opposite elements of S. Let x and y denote the projections of the centers of the spheres S_i and S_j onto l, the axis of S; then S_i and S_j are adjacent if the projection onto l of no other sphere center separates x and y. Let P be a packing of spheres into S as above defined, and let l be oriented. Then P is denoted by an ordered array of marks $S_{i1},\cdots,S_{in}$ in which adjacent marks denote adjacent spheres; and if x and y are the projections of the centers of S_{ij} and S_{ik} ($j<k$), then the orientation of the directed line segment $xy^{\rightarrow}$ is positive. The array $S_{in},\cdots,S_{i1}$ is called the reverse of P.

A series of lemmas is developed and used to prove the theorem that in the set of all possible packings of spheres into S, as above defined, and subject to the above restriction, the density is greatest for the incompressible packing

$$(1)\qquad \{\cdots S_{2i}\cdots S_6S_4S_2S_1S_3S_5S_7\cdots S_{2i+1}\cdots\};$$

and the least dense incompressible packing is

$$(2)\qquad \{S_1S_nS_3S_{n-2}S_5S_{n-4}\cdots S_{n-5}S_6S_{n-3}S_4S_{n-1}S_2\}.$$

If some of the spheres have the same radius, then there are other packings whose density is equal to those of (1) or (2). If there are no spheres with equal radii, then no other packing can have a density equal to that of (1) or (2) except their reverses. *S. Melmore* (York).

Walsh, C. E. The equal internal bisectors theorem. Edinburgh Math. Notes 37, 22–23 (1949).

Ball, Richard William. Dualities of finite projective planes. Duke Math. J. 15, 929–940 (1948).

Im Anschluss an zwei Arbeiten von R. Baer [Bull. Amer. Math. Soc. 52, 77–93 (1946); Amer. J. Math. 69, 653–684 (1947); diese Rev. 7, 387; 9, 301] leitet Verfasser mit zahlentheoretischen Hilfsmitteln Sätze her über die Anzahl N der mit ihrem Bildelement incidenten Elemente in einer Correlation d und ihren Potenzen d^i. Diese Elemente heissen bei einer Correlation absolute Elemente, bei einer Collineation Fixelemente. Jede Correlation einer endlichen Geometrie besitzt mindestens ein absolutes Element. Ferner wird bewiesen: ist e die kleinste natürliche Zahl, so dass d^e die Identität ist, und ist die ganze Zahl i in der Gruppe der Restklassen von e der zu e teilerfremden Reste enthalten, so gilt $N(d^i)=N(d)$, falls die um 1 verminderte Anzahl der Punkte auf jeder Geraden ein Quadrat ist. Ist sie kein Quadrat, so ist eine weitläufigere Fallunterscheidung erforderlich; es lassen sich eine Reihe hinreichender Bedingungen dafür angeben, dass für jedes i der genannten Art d^i ebensoviele absolute Elemente bzw. Fixelemente besitzt wie Punkte auf jeder Geraden. *R. Moufang.*

Bruins, E. M. On the symbolical method. I. Nederl. Akad. Wetensch., Proc. 51, 1270–1276 = Indagationes Math. 10, 414–420 (1948).

The author employs the symbolic method of invariant theory with the consequent fundamental identities to produce concise proofs of geometric theorems as, e.g., Desargues' theorem, Pascal's theorem. The equation to the Pascal line for six points a, b, c, d, e, f on a conic is

$$(ace)(cdf)(bdx)+(dce)(bdf)(acx)=0.$$

This is linear in a, f, b, e, but of the second degree in c, d. It is pointed out that as any of the 6 points move round the conic the Pascal line forms a pencil. Hence the expression is one "too much" in c and d. The superfluous c and d cannot be removed by using the fundamental identities because only a bracket of 3 symbols could be removed. The author therefore adopts the device of adjoining a normal curve of degree $n-1$ to a projective space of n variables. Thus in the plane the normal curve is a fixed conic with a parametric representation. A line is associated with the pair of parameters of the points of intersection with the normal curve; a point, with the pair of parameters corresponding to its polar lines. The device enables the ternary bracket factors to be expressed in terms of binary factors. Applications are made to the geometry of systems of points on a conic. *D. E. Littlewood* (Bangor).

Todd, J. A. The geometry of the binary (3, 1) form. Proc. London Math. Soc. (2) 50, 430–437 (1949).

A binary (3, 1) form is an algebraic form of degrees 3, 1 respectively in two sets of two variables, x_0, x_1; y_0, y_1, each transformed independently by full linear groups. The author has obtained in a previous paper [Proc. Cambridge Philos. Soc. 42, 196–205 (1946); these Rev. 8, 129] a complete set of 20 irreducible concomitants for this form. The present paper discusses geometric interpretations of the vanishing of these forms. Putting $X=x_0y_0$, $Y=x_0y_1$, $Z=x_1y_0$, $T=x_1y_1$, the pairs of coordinates x_0, x_1; y_0, y_1 represent the two sets of generators of the quadric Q with equation $XT-YZ=0$. The (3, 1) form f then represents a rational quartic curve C on the quadric having one set of generators as trisecants and meeting generators of the other set in single points. The properties of such a curve have been intensively studied. What is new is the interpretation of the vanishing of the concomitants of which the following examples are typical.

A set of 4 coplanar points on the curve correspond to a quartic in the parameters which is always apolar to a fixed quartic φ. Then if the invariant $I_2=0$ the fixed quartic φ

is equianharmonic. If $I_6=0$ then φ has a repeated factor. There is a concomitant $F=(f,f)^{1,1}$ of type $(4,0)$, the pair of upper suffixes indicating transvectants with respect to the x's and the y's, respectively. The equation $F=0$ gives the 4 generators (x) of Q whose intersections with C are the points where the Hessian H of φ is zero. It is known that the 6 points of C corresponding to the sextic covariant of φ form 3 mutually harmonic pairs of points; that the 3 corresponding chords are concurrent at a point O, and that their 3 polar lines lie in the polar plane π of O. Then the vanishing of $p=(f,F)^{2,0}$, a form of type $(1,1)$, gives the section of Q by the plane π. Duality between (m,n) and (n,m) forms among the concomitants the author explains as due to the fact that the given curve C on Q, of type $(3,1)$, uniquely defines another curve C_1 of type $(1,3)$ on Q with which it is in perspective from O. Finally the author considers the curves cut on Q by the quadric Σ and the Steiner surface k enveloped by planes which meet C in equianharmonic or harmonic sets. *D. E. Littlewood* (Bangor).

Hjelmslev, Johannes. On a class of biquadratic curves. Mat. Tidsskr. B. **1948**, 29–35 (1948). (Danish)

A biquadratic curve in the Euclidean or hyperbolic plane, or on the sphere, is called orthogonal if it can be generated by intersecting orthogonal circles each of which passes through two fixed points (limit circles and equidistant curves are to be counted as circles). If K_i, $i=1,2$, is a variable circle through two fixed points, and K_1 intersects K_2 at a constant angle, then the locus of the intersections is an orthogonal biquadratic curve. If two great circles on a sphere through two fixed (not antipodal) points A_1, A_2 intersect at a constant angle ν, then the locus of the intersections is a biquadratic curve, whose projections on the planes of symmetry of A_1 and A_2, A_1 and A_2', A_1' and A_2 are ellipses, where A_i' is the antipode to A_i. The eccentricity of these ellipses does not depend on ν, but only on the spherical distance of A_1 and A_2. *H. Busemann.*

Hjelmslev, Johannes. On the locus of a variable triangle. Mat. Tidsskr. B. **1948**, 36–40 (1948). (Danish)

The following result, which is essentially due to Mannheim, is proved in a simple manner. Let $A_1A_2A_3$ be a variable triangle in the Euclidean plane. If B_i is the characteristic point of the oriented line $A_{i+1}A_{i+2}=a_i$ (subscripts are to be reduced mod 3) and b_i is the direction from A_i to A_i+dA_i, then (*) $\prod_i A_{i+1}B_i \sin(a_{i+1}, b_i)=\prod_i A_{i-1}B_i \sin(a_{i-1}, b_i)$. The result is generalized to n-gons and a projective generalization is given. Also, a projective construction for the sixth piece in (*) from the other five is derived.
H. Busemann (Los Angeles, Calif.).

Satyanarayana, K. *S*-quadrics and isotomic polars. Math. Student **15** (1947), 13–15 (1948).

This paper establishes for n dimensions the generalization of the following theorem of the plane. If P and P' are isotomic conjugates with respect to a triangle, the trilinear polar of P with respect to the triangle is also the polar of P' with respect to a certain conic which passes through the vertices of the triangle. Consideration of the proof for $n=2$ shows that the theorem remains valid if P and P' are related by any quadratic transformation of the type $x_i'=a_i/x_i$, where the given triangle is the triangle of reference.
R. A. Johnson (Brooklyn, N. Y.).

Németh, E. Geometry of circle aggregates. Műegyetemi Közlemények **1948**, 57–88 (1948). (English. Hungarian summary)

The well-known cyclographic mapping is a one-to-one correspondence between the oriented circles of the plane and the points of three-dimensional Euclidean space. This correspondence may be used to derive theorems on circles. The author discusses what, according to E. Müller's terminology [E. Müller, Vorlesungen über darstellende Geometrie, v. 2, Deuticke, Wien, 1929], are called C-metric problems. The English of the paper is such that some of the theorems are unintelligible. *E. Lukacs* (China Lake, Calif.).

Woodbridge, M. Y. A geometry of clocks. Math. Mag. **22**, 129–137 (1949).

In the study of the derivative dw/dz of a polygenic function Kasner introduced the term clock to denote a circle with a directed radius. The derivative of a polygenic function is represented by a congruence of clocks. [See Science (N.S.) **66**, 581–582 (1927).] The author develops a geometry in which the fundamental elements are clocks. This is a four-dimensional geometry. The analogue of a straight line is a unial consisting of ∞^1 clocks having a common line of centers, the ends of the directed radii being collinear and these radii prolonged being concurrent. The analogue of a plane is a duplax consisting of ∞^2 clocks such that the unial determined by any two clocks of the duplax is wholly contained in it. A radiam, analogous to a three-flat, consists of the ∞^3 clocks having a common radial point. Finally the totum is the totality of all ∞^4 clocks. The interrelationships between these concepts are studied. Thus two clocks determine one and only one unial. Three clocks, not in the same unial, determine one and only one duplax. Four clocks, not in the same duplax, determine one and only one radiam. The concepts of order, parallels, congruence, continuity, and distance are developed. As these all satisfy Hilbert's axioms for Euclidean geometry, it is found that this geometry of clocks is an interpretation in the plane of Euclidean four-dimensional geometry. *J. De Cicco.*

Convex Domains, Integral Geometry

Blumenthal, Leonard M. Metric methods in linear inequalities. Duke Math. J. **15**, 955–966 (1948).

This paper deals with an extension of Minkowski's results on systems of inequalities $\sum f_i(t)\cdot x_i \geqq 0$, $i=1,\cdots,n+1$, $t\epsilon T$, when the set T is not assumed to be finite. The unit vector in E_{n+1} corresponding to $(f_1(t),\cdots,f_{n+1}(t))$ is denoted by $c(t)$ and the set of the $c(t)$ by C; C is a subset of the unit sphere $S_{n,1}$ of E_{n+1}. In the same way the set of solutions is represented by a subset S of $S_{n,1}$. The dependence of S on C is expressed by $S=\Sigma(C)$. Either there exists no closed hemisphere $H_{n,1}$ of $S_{n,1}$ containing C, and then $\Sigma(C)$ is empty, or there exists at least one supporting great sphere $S_{n-1,1}$ on $S_{n,1}$ bounding an $H_{n,1}$ containing C, and then $\Sigma(C)$ is the intersection of the $H_{n,1}$ so obtained. Therefore $\Sigma(C)$ is a closed convex subset of $S_{n,1}$. If, for a set M of $S_{n,1}$ situated on an $H_{n,1}$, we denote by M^* the least closed convex subset of $S_{n,1}$ containing M (convex closure of M), then $\Sigma(C)=\Sigma(C^*)$. A point y of M^* is called an extreme point provided no points x and z of M^* exist such that y lies between x and z; the set of the extreme points of M^* is represented by $E(M^*)$ and the convex closure of

$E(M^*)$ is proved to be equal to M^*. Hence the set $E(\Sigma C)$ appears as a complete set of fundamental solutions in the sense of Minkowski. The sets C and $\Sigma(C)$ are identical if and only if C is a closed convex subset of $S_{n,1}$ of constant breadth $\pi/2$. From a spherical theorem of Helly type due to C. V. Robinson [Amer. J. Math. **64**, 260–272 (1942); these Rev. **3**, 300] it follows that, if C has more than $2n+2$ pairwise distinct points, then the system has a nontrivial solution if and only if each subsystem of $2n+1$ inequalities has a nontrivial solution. A set C consisting of $2n+2$ points for which the preceding condition is fulfilled (pseudo-solvable system) consists of the endpoints of $n+1$ linearly independent diameters of $S_{n,1}$. [The results of the paper may be expressed by means of cones in E_{n+1} and some of them interpreted as properties of a pair of (absolutely) polar convex cones. From this point of view they are in close relation to the reviewer's note, Revue Sci. (Rev. Rose Illus.) **77**, 657–658 (1939); these Rev. **1**, 263.] *C. Y. Pauc.*

Hadwiger, H. Beweis einer Extremaleigenschaft der symmetrischen Kugelzone. Portugaliae Math. **7**, 73–85 (1948).

The paper contains a proof of the result already announced in a previous note [Hadwiger, Glur and Bieri, Experientia **4**, 304–305 (1948); these Rev. **10**, 141]. For all convex bodies of revolution in ordinary space the inequality $M\geqq(2\pi F)^{\frac{1}{2}}(\sin^2\varphi+2\cos\varphi)^{-\frac{1}{2}}(2\cos\varphi+\varphi\sin\varphi)$ is valid, where $\cos\varphi=\xi$ $(0\leqq\varphi\leqq\pi/2)$ is the only root of the equation $(3\xi-\xi^3)^2(1+2\xi-\xi^2)^{-3}=18\pi V^2F^{-3}$ in the interval $0\leqq\xi\leqq1$. The inequality is sharp in the sense that equality is attained for the symmetrical spherical zone with the same V and F.

W. Fenchel (Copenhagen).

Dinghas, A. Neuer Beweis einer verschärften Minkowskischen Ungleichung für konvexe Körper. Math. Z. **51**, 306–316 (1948).

In a previous paper, the author has associated to each convex body $\mathfrak{K}$ in Euclidean 3-space a "spherical cap body" $\mathfrak{K}^*$ [Math. Z. **47**, 669–675 (1942); these Rev. **7**, 528]. If $\mathfrak{K}$ has no three-dimensional corners, $\mathfrak{K}^*$ is equal to the unit sphere. Notations: $d\omega$ is the area element of the unit sphere; O is the surface of $\mathfrak{K}$; p is the gauge function of $\mathfrak{K}$, $M=(4\pi)^{-1}\int p d\omega$. Stars indicate the corresponding quantities for $\mathfrak{K}^*$. The author has proved the inequality (1) $M^2-OO^*\geqq0$ [Monatsh. Math. Phys. **51**, 46–56, 56a (1943); these Rev. **7**, 260], equality holding if and only if $\mathfrak{K}$ and $\mathfrak{K}^*$ are homothetic. Since $O^*\geqq4\pi$, (1) contains Minkowski's classical estimate $M^2-4\pi O\geqq0$. It improves it if $\mathfrak{K}$ has corners. In this paper a simpler proof of (1) is given. It consists essentially of two steps. (1) The Wirtinger-Blaschke lemma is proved under assumptions that are satisfied by the gauge functions of all convex bodies. This lemma states: let u be a function on the unit sphere, $\int u d\omega=0$; then $\int(2u^2-\Delta_1(u,u))d\omega\leqq0$ [cf. Math. Z. **47**, 265–274 (1941); these Rev. **3**, 300; Δ_1 is Beltrami's first differential operator]. (2) Application of this lemma to the function $u=p/M-p^*/M^*$ reduces the discussion of (1) to the proof of $M=\int(pp^*-\frac{1}{2}\Delta_1(p,p^*))d\omega$.

P. Scherk (Saskatoon, Sask.).

Dinghas, A. Neuer Beweis einer isoperimetrischen Ungleichung von Bol. Math. Z. **51**, 469–473 (1948).

A proof, based on Wirtinger's inequality [see Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, 7.7.4, p. 185], of an isoperimetric inequality $L^2\geqq4FF^*$ due to Bol [Nieuw Arch. Wiskunde (2) **20**, 171–175 (1940); these Rev. **1**, 158], which concerns the length L of a plane convex curve C, the area F interior to C and the area F^* of the unit circle augmented by "caps" parallel to the corners of C. A linear inequality connecting L, F, F^* is also treated. *L. C. Young* (Madison, Wis.).

Schmidt, Erhard. Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie. I. Math. Nachr. **1**, 81–157 (1948).

Lusternik proved the Brunn-Minkowski theorem for sets in Euclidean n-space that are not necessarily convex [C. R. (Doklady) Acad. Sci. URSS. (N.S.) **8(1935 III)**, 55–58]. In this paper, a self-contained elementary proof is given in which the Euclidean case is treated simultaneously with the spherical and hyperbolic ones. Let R_n denote an n-space that is either Euclidean, hyperbolic or spherical. Hyperbolic R_n may be represented by Poincaré's model, while the spherical R_n of curvature one may be identified with the surface of the unit sphere in Euclidean R_n.

In the following, $\mathfrak{K}$, $\mathfrak{K}_1$, $\mathfrak{K}_2$ denote closed bounded nonempty point sets in R_n; $\bar{\mathfrak{K}}$ is the supplementary set of $\mathfrak{K}$; $\delta\{\mathfrak{K},\mathfrak{H}\}$ and $D\{\mathfrak{K},\mathfrak{H}\}$ denote the greatest lower and least upper bounds of the distances between the points of $\mathfrak{K}$ and those of another set $\mathfrak{H}$. Furthermore, $\mathfrak{K}_h$ is the set of all points P with $\delta\{\mathfrak{K},P\}\leqq h$, $\mathfrak{K}^{(h)}$ the set of all P with $D\{\mathfrak{K},P\}\leqq h$ ($0<h<\pi$ in the spherical case, $0<h$ otherwise); $V(\mathfrak{K})$, $\cdots$, denote the n-dimensional Lebesgue volume of $\mathfrak{K}$, $\cdots$; r, r_h, $r^{(h)}$, $\cdots$, the radii of (solid) spheres in R_n of volumes $V(\mathfrak{K})$, $V(\mathfrak{K}_h)$, $V(\mathfrak{K}^{(h)})$, $\cdots$.

Brunn-Minkowski theorem. First form: If $\bar{\mathfrak{K}}_h$ is not empty, then (1) $f(h)=r_h-(h+r)\geqq0$, equality holding if and only if $\mathfrak{K}$ is a sphere. Second form: If $\mathfrak{K}_1\subset\mathfrak{K}_2$ and if $\bar{\mathfrak{K}}_2$ is not empty, then (2) $r_2-r_1\geqq\delta\{\mathfrak{K}_1,\bar{\mathfrak{K}}_2\}$. The condition for equality is similar to that for (4)

"Spiegeltheorem." First form: If $\mathfrak{K}^{(h)}$ is not empty, then (3) $g(h)=(h-r)-r^{(h)}\geqq0$. The condition for equality is the same as for (1). Second form: Suppose $D\{\mathfrak{K}_1,\mathfrak{K}_2\}<\pi$ in the spherical case; then (4) $r_1+r_2\leqq D\{\mathfrak{K}_1,\mathfrak{K}_2\}$ in all three geometries, equality holding if and only if $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are concentric spheres.

Both forms of either theorem are equivalent. Furthermore, both theorems are equivalent in spherical geometry, since then $f(h)=g(\pi-h)$. The Brunn-Minkowski theorem implies the isoperimetric inequality (5) $O(\mathfrak{K})\geqq O(\mathfrak{K}_0)$, including the condition for equality. Here $\mathfrak{K}_0$ is the sphere of radius r and $O(\mathfrak{K})=\liminf_{h\to0}(V(\mathfrak{K}_h)-V(\mathfrak{K}))/h$ and $O(\mathfrak{K}_0)$ are the surfaces of $\mathfrak{K}$ and $\mathfrak{K}_0$.

The main tool in the proof of (2) and (4) is a "rotational symmetrization." Through a point O construct an $(n-1)$-plane E and a straight line perpendicular to E. Let $E(t)$ be the locus of all points of distance t from E, this distance being measured with opposite signs on both sides of E. If $\mathfrak{K}\cdot E(t)$ is not empty, construct the $(n-1)$-sphere in E with center O and with an $(n-1)$-dimensional volume equal to that of the perpendicular projection of $\mathfrak{K}\cdot E(t)$ on E, and let $\mathfrak{K}^*(t)$ be the point set in $E(t)$ whose perpendicular projection on E is equal to this $(n-1)$-sphere. Then the union $\mathfrak{K}^*$ of all $\mathfrak{K}^*(t)$ is again closed, bounded, and nonempty, and we have $r^*=r$. Suppose (2) and (4), true for $n=1$, have been proved up to $n-1$. Construct $\mathfrak{K}_1^*$ and $\mathfrak{K}_2^*$ out of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ by the same rotational symmetrization. Then the induction assumption implies that $\delta\{\mathfrak{K}_1^*,\bar{\mathfrak{K}}_2^*\}\geqq\delta\{\mathfrak{K}_1,\bar{\mathfrak{K}}_2\}$, or $D\{\mathfrak{K}_1^*,\mathfrak{K}_2^*\}\leqq D\{\mathfrak{K}_1,\mathfrak{K}_2\}$, respectively. Hence it suffices

to prove (2) and (4) for $\mathfrak{K}_1^*$ and $\mathfrak{K}_2^*$. This is essentially a two-dimensional problem. Its solution is by far the most intricate part of this proof, however.

In the last paragraph the monotonicity of the continuous functions $f(h)$ and $g(h)$ is proved. Let h_0' denote the smallest h' for which $\mathfrak{K}^{(h')}$ is not empty. Then, in the spherical case $0\leqq f(h_1)\leqq f(h_2)\leqq f(h_0)=\pi-(r+h_0)$ if $0<h_1<h_2<h_0=\pi-h_0'$, and in the other two geometries

$$0\leqq f(h_1)\leqq f(h_2)\leqq \lim_{h\to\infty} f(h)\leqq \lim_{h'\to\infty} g(h') \leqq g(h_2')\leqq g(h_1')\leqq g(h_0')=h_0'-r$$

if $0<h_1<h_2$, $h_0'<h_1'<h_2'$. These results reduce the discussion of the equality sign in (1) and (3) to that of (5). This discussion is to be carried through in the second part of this paper which will also contain linear improvements of (1), (3) and (5). *P. Scherk* (Saskatoon, Sask.).

Schmidt, Erhard. Der Brunn-Minkowskische Satz und sein Spiegeltheorem sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und hyperbolischen Geometrie. Math. Ann. 120, 307–422 (1948).

In the paper reviewed above, the topology of spherical R_n makes for considerable complications. Thus major simplifications can be obtained immediately if the spherical case is omitted. This holds true in particular for the proof of the inequalities (1)–(4) [cf. the preceding review]. The proofs of these relations in the present paper are essentially the same as those arrived at in the indicated fashion. In addition, this paper develops at length the subject matter mentioned at the end of the preceding review. [This paper was written before the one reviewed above.] *P. Scherk.*

Blaschke, Wilhelm. Über dichtetreue Geradenabbildungen in der Ebene. Arch. Math. 1, 234 (1948).

Let p be a fixed point in the Euclidean plane, h the distance of a line q from p and ϕ the angle of q with a fixed direction. Then $dhd\phi$ measures density of lines. The density is invariant under motions of the plane. A simple example is given to show that there are other mappings of the set of lines in the plane on itself which preserve density. *H. Busemann* (Los Angeles, Calif.).

Algebraic Geometry

Wiman, A. Über rationale Punkte auf Kurven dritter Ordnung vom Geschlechte Eins. Acta Math. 80, 223–257 (1948).

In a previous paper the author has studied the rank r of the elliptic cubics (I) $y^2=x(x+a)(x+b)$, where a and b are rational [Acta Math. 76, 225–251 (1945); these Rev. 7, 70]; then, in another paper, he has applied his methods to the harmonic cubics (II) $y^2=x(x^2-c^2)$ [Acta Math. 77, 281–320 (1945); these Rev. 7, 323]. He now obtains several further cubics, of the form (I) and (II), having the rank not less than 6 or 5 respectively. For this purpose he uses again the methods explained in the former work; but the difficulty of the discussion necessary for finding the new examples is much increased, and shows the extreme intricacy of the still unsolved problem of determining whether r is bounded or not and of finding, in the affirmative case, an upper bound for it.

The paper is divided into three parts. Part I deals with the rank of the families of curves obtained by the author's methods, and also considers the different ways of writing the equation of a given cubic in form (I); to the former end certain assumptions are made which, though probable, seem rather difficult to prove. Part II is concerned with the equations (II). The author, in the second paper quoted above, has shown that $r\geqq 6$ if $c=2\cdot3\cdot7\cdot17\cdot41$; he is now doubtful whether there are other curves (II) having $r\geqq 6$, but obtains several cases in which $r\geqq 5$, i.e.,

$$\begin{aligned} c=2\cdot3\cdot5\cdot7\cdot13\cdot17\cdot19,&\quad 2\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19,\\ 2\cdot3\cdot5\cdot7\cdot11\cdot19\cdot23,&\quad 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot19\cdot23,\\ 2\cdot3\cdot5\cdot7\cdot19\cdot41.& \end{aligned}$$

In the first paper quoted the author obtained the curve (I) with $a=46$, $b=292$ and proved that it has $r\geqq 6$. In part III he adds many more curves having $r\geqq 6$, namely the curves (I) with

$$\begin{aligned}(a, b)=&(9\cdot61,\ 12\cdot283),\quad (18\cdot53,\ 12\cdot419),\\ &(-2^6\cdot185\cdot17\cdot23,\ 2\cdot185\cdot5\cdot641),\\ &(77\cdot17\cdot64,\ 77\cdot11\cdot547),\\ &(-14\cdot13\cdot25\cdot157,\ 14\cdot13\cdot31\cdot112),\\ &(7\cdot29\cdot512,\ -49\cdot167).\end{aligned}$$

He also shows that the rank of the general curve of one of his families may be lower than one would expect. *B. Segre* (Bologna).

Chow, Wei-Liang. On the genus of curves of an algebraic system. Trans. Amer. Math. Soc. 65, 137–140 (1949).

A new proof of the theorem that, if $\{C\}$ is an irreducible algebraic system of curves in S_n, whose generic member is irreducible and of genus g, and if C_0 is any irreducible member of $\{C\}$, then the genus of C_0 cannot exceed g. The method is to show that there is an infinite number of integers n such that on C_0 there are linear series of order n and dimension $n-g$ at least. The associated (zugeordnete) form is used to show that the condition for the existence of a series of order n and dimension $n-g$ at least is an algebraic one (on the parameters of $\{C\}$), and the result follows immediately by use of the theory of algebraic correspondences. *W. V. D. Hodge* (Cambridge, England).

Semple, J. G. On complete quadrics. I. J. London Math. Soc. 23, 258–267 (1948).

Study has shown that all the complete conics of the plane, i.e., conics completely specified both as loci and as envelopes, can be represented without exception on the points of an algebraic manifold $M_5^{102}[27]$; van der Waerden has shown that M is free from multiple points, while Severi has noted that the two primary degeneration manifolds are also nonsingular and intersect without contact in the secondary degeneration manifold. The author considers the analogous problem for quadrics. Let Q_L, Q_E and Q_C denote the quadric locus, the quadric envelope and the quadratic line-complex, respectively, the coefficients of the equations being a_{rs}, b_{rs}, $c_{rr',ss'}$. To fix a basis for a formal definition of "complete quadric," covering cases in which some or all of Q_L, Q_E, Q_C are degenerate, he selects a set of relations (A), which are sufficient to characterize the relations between Q_L, Q_E, Q_C when Q_L is generic and the other two derived from them and he says then that any triad constitutes a complete quadric if its sets of coefficients satisfy the relations (A). These relations are bilinear between a and b, a and c, b and c. By transformation of the coordinate system he proves that (A) admits as solutions all the types of complete quadric that are known geometrically and that

they admit no other solutions. There are 8 types; (I) is the general quadric; and for (V), for example, we have: Q_L is a plane π^2, Q_S is a conic k in π, Q_C consists of the secants of k. To obtain a model of all complete quadrics the totality of points is considered whose homogeneous coordinates are given by $a_{r_1s_1}$, $b_{r_2s_2}$, $c_{r_3r'_3, s_3s'_3}$, this giving a Segre variety, product of two [9]'s and a [20].

Extending the method of van der Waerden the author gives a parametric representation of a complete quadric, observing that the equation of the generic Q_L can be expressed rationally, and in one way only, in the form

$$X_1^2+q_2X_2^2+q_2q_3X_3^2+q_2q_3q_4X_4^2=0,$$
$$X_1=x_1+m_{12}x_2+m_{13}x_3+m_{14}x_4,\quad X_2=x_2+m_{23}x_3+m_{24}x_4,$$
$$X_3=x_3+m_{34}x_4,\quad X_4=x_4,$$

so that Q_L is birationally represented by the points of S_9 with the nonhomogenous coordinates (q_i, m_{jk}). The three primary degeneration manifolds are represented by the primes $q_4=0$, $q_3=0$ and $q_2=0$ respectively. The author shows by means of this representation that his manifold M_9 is free from multiple points and that the subordinate manifolds of M_9 (three of dimension 8, three of dimension 7 and one of dimension 6) which represent degenerate complete quadrics are all likewise nonsingular; the first three of them intersect simply in the other four. *O. Bottema.*

Abellanas, Pedro. Projective and algebraic spaces. Revista Acad. Ci. Zaragoza (2) **3**, 11–17 (1948). (Spanish)
Expository lecture.

Gaeta, Federico. On the classification of the algebraic curves of an S_r. Revista Mat. Hisp.-Amer. (4) **8**, 165–173 (1948). (Spanish)

Étant données deux courbes complémentaires, irréductibles et sans points multiples, d'un S_r ($r\geqq 3$), d'ordres N, N' et de genres p, p', constituant l'intersection complète de $r-1$ formes d'ordres $n_1, n_2, \cdots, n_{r-1}$, on sait que les formes d'ordre $l\geqq\rho=\sum_{i=1}^{r-1}n_i-r-1$ déterminent sur C une série linéaire complète. L'auteur établit le théorème suivant. Si Δ_l, Δ_l' sont les déficiences des séries linéaires que les formes d'ordre $l<\rho$ de S_r déterminent respectivement sur C et C', on a: $\Delta_l=\Delta'_{\rho-l}$, $\rho=\sum n_i-r-1$. Il rattache au théorème précédent la propriété suivante relative aux courbes arithmétiquement normales établie dans un travail antérieur [Revista Mat. Hisp.-Amer. (4) **7**, 255–268 (1947); ces Rev. **9**, 526]: si C est arithmétiquement normale, C' l'est aussi. Pour les couples de surfaces régulières (irréductibles et sans points multiples) F et F' qui constituent l'intersection complète et simple de $r-2$ formes d'ordres $n_1, n_2, \cdots, n_{r-2}$ de S_r ($r\geqq 4$), il établit la proposition suivante, analogue à celle relative aux couples de courbes (C, C'): si Δ_l et Δ_l' sont les déficiences des systèmes linéaires que les formes d'ordre $l<\rho=\sum_{i=1}^{r-2}n_i-r-1$ déterminent sur F et F', on a $\Delta_l=\Delta'_{\rho-l}$. Il montre ensuite comment la théorie des séries linéaires fournit la possibilité d'établir une classification des courbes algébriques irréductibles sans points multiples de S_r ($r\geqq 3$). *P. Vincensini* (Besançon).

Lage Sundet, Knut. On the tangents in multiple points of algebraic curves. Norsk Mat. Tidsskr. **30**, 65–68 (1948).

It is shewn that all irreducible curves of order n having certain fixed (usually multiple) points and certain fixed tangents in these, have in common also the remaining tangents in these points, under the following conditions: n has a factor t, satisfying $n\geqq 3t(t-1)$, and the orders of multiplicity of the points in question are $a_f=b_f\cdot t/n$ ($f=1, 2, \cdots, i$), where $\sum b_f=3(t-1)$, $\sum b_f^2=t^2-1$. The number of tangents in the multiple points determined by the rest is then $\frac{1}{2}(3t-4)(3t-5)$. The method involves the Cremona transformation of order t with base points of multiplicities $b_1, \cdots, b_i$ at the points in question.

In the case $t=2$, $b_f=1$ ($f=1, 2, 3$), all $2n$-ics having three fixed n-ple points and all but one of the tangents at these in common, have also in common the $3n$th tangent. It is also shewn by a dual transformation that all curves of class n touching all the fixed tangents in one point, all but one of those in another, and the harmonic conjugates of those in the third with respect to the lines joining this point to the other two fixed multiple points, likewise touch the remaining tangent in the second point. *P. Du Val.*

Galafassi, Vittorio Emanuele. I tactinvarianti nella topologia dello spazio proiettivo. Boll. Un. Mat. Ital. (3) **3**, 18–25 (1948).

Let F^a be an algebraic surface of degree a in the 3-dimensional projective space, let T be the tactinvariant of F^m and F^n, and T' the tactinvariant of F^{n_1}, F^{n_2} and F^{n_3}. Then T vanishes if and only if F^m touches F^n; T' vanishes if and only if one of the surfaces F^{n_1}, F^{n_2} and F^{n_3} touches the complete intersection of the other two. If the surfaces under consideration are real, T and T' are real as well. The author finds a topological interpretation of the sign of T', and also of the sign of T if $m-n$ is even. He proves that if $m-n$ is odd, the sign of T has no topological meaning. *F. J. Terpstra* (Bandung).

Enriques, F. Sur la démonstration géométrique d'un théorème de Picard, concernant les surfaces algébriques. Revista Acad. Ci. Madrid **42**, 5–7 (1948).

The author proves the following lemma. If C is an irreducible algebraic curve without singularities on an irregular surface F, then it is always possible to construct an irreducible regular linear system $|D|$, of arbitrarily large freedom, which contains C and is such that the linear series cut by $|D-C|$ on a curve of $|D|$ is not complete. The system $|D|$ may, for instance, be any suitably large multiple of the linear system of prime sections of F. This lemma is in contradiction with one stated by Severi [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **17**$_2$, 465–470 (1908)] and used by him to deduce an algebro-geometric proof of the theorem that the linear system $|C'|$ adjoint to any irreducible curve C on a surface which belongs to a continuous system of positive degree is regular (a result proved transcendentally, with more restrictions on C, by Picard). In view of Enriques' result this proof appears to require reconsideration. The error in Severi's argument is indicated by Enriques at the end of the paper. *J. A. Todd* (Cambridge, England).

Please see Errata, p. 870, these Rev. 11, 1950

Jongmans, F., et Nollet, L. Un théorème sur les systèmes linéaires de courbes algébriques planes à système adjoint réductible. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 617–625 (1948).

The authors remark that the theorem stated by S. Kantor [Acta Math. **19**, 115–193 (1895)] to the effect that, if the pure adjoint system $|C'|$ of an infinite linear system $|C|$ of plane curves is reducible, it is compounded of a pencil of rational curves, is not completely proved. In a recent paper [Mém. Soc. Roy. Sci. Liége (4) **7**, 469–554 (1947); these Rev. **9**, 461] Nollet has shown that the effective genus of the curves of the pencil $|E|$ of which $|C'|$ is compounded

is less than 3. In the present note it is shown that $|E|$ cannot consist of elliptic curves. The possibility of the curves $|E|$ being of genus 2 is left open; if this case can in fact arise then $|C|$ must be a pencil of hyperelliptic curves.
J. A. Todd (Cambridge, England).

Jongmans, F. Les générations d'une surface algébrique au moyen de deux réseaux réciproques de surfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 754–762 (1948).

The author's problem is essentially the following. It is required to express the equation of the generic surface of order n in space in the form $P_1Q_1+P_2Q_2+P_3Q_3=0$, where P_1, P_2, P_3 are forms of order p, and Q_1, Q_2, Q_3 are forms of order q ($p\leqq q$, $p+q=n$). It is shown that such an expression is impossible if $p>7$ except in the case $p=q=8$, and is always possible if $p=q=8$, $n=16$ or if $p\leqq 7$, $n\geqq 4p-3$. The remaining cases are undecided. [Reviewer's note. There are a number of confusing misprints concerning inequality signs.] *J. A. Todd* (Cambridge, England).

Nollet, Louis. Quelques relations entre les invariants numériques d'une surface algébrique qui entraînent l'existence, sur la surface, d'un faisceau irrationnel de courbes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 763–771 (1948).

If F is an algebraic surface of positive irregularity q such that $p^{(1)}\leqq 4p_a$, then either F contains a pencil of genus q of curves each containing a g_2^1 or a g_3^1, or else $p_a\geqq 2$, $p^{(1)}>3p_a+1$. *J. A. Todd* (Cambridge, England).

Godeaux, Lucien. Sur les droites appartenant à une surface à sections elliptiques. Simon Stevin **26**, 12–14 (1948).

L'auteur démontre pour les surfaces normales à sections elliptiques des espaces à quatre, cinq ou six dimensions des propriétés analogues à celles des surfaces cubiques de l'espace ordinaire, en utilisant dans ce but la représentation plane de ces surfaces. Il établit les théorèmes suivants. (1) Les seize droites appartenant à la surface intersection de deux hyperquadriques de l'espace à quatre dimensions forment l'intersection complète de cette surface et d'une hypersurface du quatrième ordre. (2) Les dix droites de la surface normale d'ordre cinq, à sections elliptiques, de l'espace à cinq dimensions forment l'intersection complète de cette surface et d'une hyperquadrique. (3) Les dix droites de la surface normale du sixième ordre à sections elliptiques de l'espace à six dimensions, forment l'intersection complète de cette surface et d'un hyperplan. *M. Piazzolla-Beloch.*

Godeaux, Lucien. Remarques sur les surfaces algébriques possédant une involution cyclique privée de points unis. Ann. Soc. Polon. Math. **20** (1947), 241–250 (1948).

(1) Si une surface régulière F, de genre arithmétique $p_a>1$ et dont les systèmes pluricanoniques ne sont pas composés au moyen d'un faisceau contient une involution cyclique d'ordre premier p, privée de points unis et dont l'image est une surface φ de genre arithmétique $p_a'>0$, le système canonique $|C|$ de F contient p systèmes linéaires partiels, appartenant à l'involution; l'un a la dimension $p_a'-1$ et est le transformé du système canonique de φ; les autres ont la dimension p_a'. (2) Dans les mêmes conditions si $p_a'=0$, le système canonique $|C|$ de F contient $p-1$ courbes isolées, appartenant à l'involution. On a $p_a=p-1$. (3) Si une surface E d'irrégularité q, de genre arithmétique $p_a>1$, dont les systèmes pluricanoniques ne sont pas composés au moyen d'un faisceau, contient une involution cyclique d'ordre premier p, privée de points unis, et dont l'image est une surface φ d'irrégularité q' et de genre arithmétique $p_a'>1$, le système canonique de F contient $\nu=p$ systèmes linéaires partiels appartenant à l'involution, dont l'un est le transformé du système canonique de Φ. Si $p_a'=0$, on a $\nu=p-1$. *M Piazzolla-Beloch* (Ferrara).

Godeaux, Lucien. Sur la construction de surfaces doubles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 198–205 (1948).

L'auteur considère une surface cubique possédant quatre points doubles coniques. Cette surface représente l'involution du second ordre engendrée dans un plan par une transformation quadrique involutive. L'auteur considère une transformée rationelle ϕ_0 de la surface cubique, obtenue en faisant correspondre, aux plans de l'espace, des surfaces d'ordre n d'un système linéaire triplement infini tout-à-fait général. La surface ϕ_0 est d'ordre $3n$, ayant $4n^2$ points doubles coniques. Elle corresponde, comme l'auteur démontre, aux conditions nécessaires et suffisantes pour qu'une surface algébrique réprésente une involution d'ordre deux, n'ayant qu'un nombre fini de points unis, appartenant à une surface algébrique. La surface ϕ_0 est donc image d'une involution du second ordre appartenant à une surface F_0 (et possédant $4n^2$ points unis). Les points de diramation sur la surface ϕ_0 sont les $4n^2$ points doubles coniques de ϕ_0. La surface ϕ_0 est régulière; son genre arithmétique est $p_a^1=\frac{1}{2}(n-1)(3n-1)(3n-2)$, et son genre linéaire $3n(3n-4)^2+1$. Le genre arithmétique de F_0 est $p_a=(u-1)(8n^2-10n+1)$ et son genre linéaire $p^{(1)}=6n(3n-4)^2+1$. *M. Piazzolla-Beloch* (Ferrara).

Godeaux, Lucien. Recherches sur les points unis isolés des involutions cycliques appartenant à une surface algébrique. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 206–228 (1948).

L'auteur reprend les considérations sur la structure des points unis d'une involution cyclique d'ordre premier p n'ayant qu'un nombre fini de points unis, appartenant à une surface algébrique, donnant plus de généralité à une méthode esquissée dans une note antérieure [Bull. Soc. Roy. Sci. Liége **10**, 290–295 (1941); ces Rev. **7**, 28]. A la fin l'auteur applique la methode à certains cas simples, c'est-à-dire aux premières valeurs de p. *M. Piazzolla-Beloch* (Ferrara).

Godeaux, Lucien. Recherches sur les points unis isolés des involutions cycliques appartenant à une surface algébrique. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 290–302 (1948).

L'auteur complète les considérations de la note I [voir l'analyse ci-dessus], dans une hypothèse particulière.
M. Piazzolla-Beloch (Ferrara).

Godeaux, Lucien. Sur une involution cyclique du onzième ordre appartenant à une surface algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 303–313 (1948).

Dans cette note l'auteur utilise des résultats antérieurs [voir les deux analyses ci-dessus], en étudiant la structure des points unis dans le cas $p=11$. *M. Piazzolla-Beloch.*

Godeaux, Lucien. Sur les points unis des involutions cycliques appartenant à une variété algébrique à trois dimensions. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 419–425 (1948).

L'auteur considère une variété algébrique V, à trois dimensions, contenant une involution cyclique I_p d'ordre premier p, n'ayant qu'un nombre fini de points unis. Soient

A un de ces points unis, α l'espace linéaire à trois dimensions tangent en A à la variété V. Dans la gerbe de droites de sommet A dans l'espace α, la transformation birationelle T de V en soi, génératrice de l'involution I_p, détermine une homographie h. Celle-ci peut être (1) l'identité (point uni de première espèce); (2) une homologie (point uni de seconde espèce); (3) une homographie non homologique (point uni de troisième espèce). Dans cette première note l'auteur étudie la structure des points unis de première espèce et les points de diramation correspondants sur la variété image de l'involution. Il démontre que en un point de diramation correspondant à un point uni de première espèce, la variété image d'une involution cyclique d'ordre p appartenant à une variété algébrique à trois dimensions, possède un point multiple d'ordre p^2. Les sections hyperplanes du cône tangent sont des surfaces représentant les courbes d'ordre p d'un plan, ou des projections de ces surfaces.

M. Piazzolla-Beloch (Ferrara).

Godeaux, Lucien. Sur les points unis des involutions cycliques appartenant à une variété algébrique à trois dimensions. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 518–530 (1948).

Dans cette seconde note [voir l'analyse ci-dessus] l'auteur commence l'étude des points unis de seconde espèce et démontre le théorème suivant. Si une variété algébrique à trois dimensions contient une involution cyclique d'ordre premier $p=2\nu+1$, possédant un point uni de seconde espèce, on peut prendre comme image de cette involution une variété normale à trois dimensions, sur laquelle le point de diramation correspondant peut être un point multiple d'ordre $2\nu(\nu+1)+1$, le cône tangent en cè point à la variété étant formé d'un espace linéaire à trois dimensions et d'un cône rationel à trois dimensions d'ordre $2\nu(\nu+1)$, rencontrant l'espace tangent suivant un plan.

M. Piazzolla-Beloch (Ferrara).

Godeaux, Lucien. Points unis isolés des involutions cycliques appartenant à une surface algébrique. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 243–246 (1948).

Godeaux, Lucien. Points unis symétriques des involutions cycliques appartenant à une surface algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 843–845 (1948).

Godeaux, Lucien. Sur les systèmes linéaires de courbes planes ayant pour adjoint un faisceau de cubiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 846–850 (1948).

De Bono, Assunta. Trasformazioni cremoniane reali tra piani con speciale riguardo a quelle di ordine $\leqslant 5$. Boll. Un. Mat. Ital. (3) **3**, 128–135 (1948).

Bericht über die Dissertation des Verf. [Pavia, 1946]. Verf. klassifiziert die reellen Cremona-Transformationen mit lauter verschiedenen Fundamentalpunkten der Ordnungen 2, 3, 4, 5 zunächst nach den Vielfachheiten der Fundamentalpunkte, dann danach, welche Fundamentalpunkte reell sind, welche paarweise konjugiert komplex sind. Dann wird durch die reellen Fundamentalkurven die projektive Ebene in verschiedene Gebiete zerschnitten, und die Lage dieser Gebiete zu einander sowie die Lage der Fundamentalpunkte in diesen Gebieten ergibt eine weitere Möglichkeit der Unterscheidung in zahlreiche Typen. Verf. untersucht jedesmal den Typus der inversen Transformation.

O.-H. Keller (Dresden).

Casadio, Giuseppina. Sulle direzioni caratteristiche delle trasformazioni cremoniane determinate da una rigata cubica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **10(79)**, 201–208 (1946).

A space Cremona transformation [3, 3] has its homaloidal systems constituted by cubic ruled surfaces having a double line, two simple lines and two points in common. In this paper it is proved that the inflexional directions through a point P are those of: (1) the pencil of lines passing through P and intersecting the double line; (2) two lines in the plane through P and the two common points. Some indications concerning the inflexional directions are also given for a Cremona transformation of type [3, 4] or [3, 5].

E. Bompiani (Pittsburgh, Pa.).

Differential Geometry

Colmez, J. Sur certains systèmes triples orthogonaux paratingents. Ann. Sci. École Norm. Sup. (3) **65**, 71–99 (1948).

This memoir represents a continuation and an extension of Llensa's study [thesis, Paris, 1947; see partial accounts in Bull. Sci. Math. (2) **65**, 225–250 (1941); C. R. Acad. Sci. Paris **220**, 297–298 (1945); **222**, 845–847 (1946); these Rev. **7**, 77, 173, 481] of triply orthogonal systems from the point of view of Bouligand's direct infinitesimal geometry. The triply orthogonal systems considered here consist of the level surfaces of three functions $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ defined on a bounded domain (open and simply connected set) of E_3 and possessing continuous, nonvanishing, mutually orthogonal gradients. The Darboux-Lamé equation used by Llensa is replaced by the more general $(\mathcal{E})$: $|\text{grad } u| = F(M, u)$, where $F(M, u)$ is a positive function defined on a domain R of the space (x, y, z, u), continuous in u and possessing continuous second derivatives with respect to x, y, z. A solution u of $(\mathcal{E})$ provided with continuous first derivatives in x, y, z is called a paratingent integral. Any section $u=u_0$ of a paratingent integral is of bounded curvature (superderivability phenomenon); conversely every surface of bounded curvature in $R(u_0)$ (section of R by $u=u_0$) is a u_0-section of a paratingent integral which is uniquely defined. If the level surfaces Σ_{u_0} (in E_3) of a paratingent integral are imbeddable in a paratingent triply orthogonal system, they possess a normal derivable with respect to v and w. Bouligand's results on the orthogonal trajectories of the surfaces of one system in a paratingent triply orthogonal system when the latter are planes are extended to "regular" systems, the regularity being expressed by derivability conditions. Conversely, regularity assumptions on the orthogonal trajectories imply boundedness of the curvature of the corresponding surfaces. The paper ends with a new generalization of Dupin's theorem.

In the study of equation $(\mathcal{E})$, classical methods used in the theory of Hamilton-Jacobi are transposed, for the case of a circular indicatrix. Transversals are here orthogonal trajectories of the Σ_u; they are characterized among the curves with continuous tangent as minimizing for $u>u_0$ the solutions of the equation $dT/ds=F(M(s), T)$ assuming on a fixed Σ_{u_0} the value u_0. The solutions with spherical (in the Finsler sense) level surfaces in the theory of Hamilton-Jacobi are introduced under the name of semi-paratingent integrals; they constitute a complete solution of $(\mathcal{E})$.

Definitions, proofs and assumptions are carefully worked out; some of them are involved, e.g., the regularity conditions mentioned above. The topological questions connected with a global investigation of the solutions of $(\mathcal{S})$ are not tackled. *C. Y. Pauc* (Cape Town).

Barbilian, D. Bemerkung über das "Theorema egregium." Bull. Math. Phys. Éc. Polytech. Bucarest **10** (1938–39), 42–46 (1940).

Let (x, x) denote the absolute quadratic canonical form (in the sense of Cayley) of the hyperbolic space $\mathfrak{H}$. In what follows the homogeneous coordinates x_0, x_1, x_2, x_3 of any point in $\mathfrak{H}$ are assumed to be normalized by the condition $(x, x) = -k^2$, where $-1:k^2$ represents the Gaussian curvature of the space; consequently for a vector dx, we have $ds^2 = \|dx\|^2 = (dx, dx)$. On a surface $\mathfrak{S}$ in $\mathfrak{H}$ defined parametrically by $x_j = x_j(u, v)$, $j = 0, 1, 2, 3$, ds^2 is a quadratic form $Edu^2 + 2Fdudv + Gdv^2$ expressible as a product of (complex) Pfaffians $\omega = Adu + Bdv$, $\pi = Cdu + Ddv$, the general form of the factors being $e^P \cdot \omega$ and $e^{-P} \cdot \omega$ $(P = P(u, v))$. An expression in A, B, C, D and their derivatives with respect to u and v is produced, the value K of which is invariant through every change of both parameter and factorization of ds^2; in case $ds^2 = 2\bar{F}d\bar{u}d\bar{v}$ (isotropic parametrization), $K = -(\log \bar{F})_{\bar{u}\bar{v}} : \bar{F}$. If y denotes the absolute pole of the tangent plane in x to $\mathfrak{S}$, the coordinates being normalized by $(y, y) = k^2$, the second fundamental form $Ldu^2 + 2Mdudv + Ndv^2$ is defined as $(y, d^2x) = -(dy, dx)$. The invariant $k^{-2}(LN - M^2)/(EG - F^2) = Q$ is proved to be equal to $-(\log \bar{F})_{\bar{u}\bar{v}}/F + 1/k^2$. Finally $K = Q - 1/k^2$, which expresses the theorema egregium in the hyperbolic case.

C. Y. Pauc (Cape Town).

This paper was reviewed from a reprint. The bibliographical data should be (6) 6 (1943-49), no 2, 17 pp (1950)

*Salini, Ugo. **Le sviluppanti $S^{(1)}_{\varphi_1}$ ed $S^{(2)}_{\varphi_1}$ di una curva del piano proiettivo.** Atti Accad. Gioenia Catania (6) **6**, 17 pp. (1942).

Salini, U. Le sviluppanti $S^{(1)}_{\varphi_2}$ ed $S^{(2)}_{\varphi_2}$ di una curva del piano proiettivo. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 240–245 (1946).

Salini, U. Le sviluppanti $S^{(1)}_{\varphi_3}$ ed $S^{(2)}_{\varphi_3}$ di una curva del piano proiettivo. Atti Accad. Peloritana. Cl. Sci. Fis. Mat. Nat. (4) **2**(47), 19 pp. (1946).

These papers concern themselves with envelopes of certain lines associated with a plane curve C. The first may be summarized as follows. Let Γ_λ be the pencil of nodal cubics with node at a point O of C and having seven-point contact with C at O. The line of inflexions is known to be tangent to the osculating conic of C at O at its intersection P with the nodal tangent t_λ of Γ_λ not tangent to C at O. It is proposed first so to determine the parameter λ of the pencil Γ_λ that the locus $S^{(1)}$ of P as O generates C is enveloped by t_λ. There is a one-parameter family of such curves $S^{(1)}(C)$. These have the property that the tangents to any particular four of them with parameter values (c_1, c_2, c_3, c_4) cut C in points O_1, O_2, O_3, O_4 with parameter values (t_1, t_2, t_3, t_4) such that the cross ratios (c_1, c_2, c_3, c_4), (t_1, t_2, t_3, t_4) are equal. The differential equations of C being in the form $d^3x/dt^3 + rx = 0$, the parameter t has geometrical significance, and moreover the differential equation is unchanged in form by the transformation $\bar{t} = (\alpha t + \beta)/(\gamma t + \delta)$, $\alpha\delta - \beta\gamma \neq 0$. It is suggested that this property may be used as a means of defining geometrically the cross ratio of four points on a plane curve. The line of inflexions l_λ cuts the tangent to C at O in a point N. The parameter λ is now determined so that the locus $S^{(2)}$ of N is enveloped by l_λ. This condition on λ is the same as for the discussion above. That is, if P generates a curve tangent to t_λ, then N generates a curve tangent to l_λ as O generates C.

These ideas are now generalized by replacing the osculating conic of C by its transform $\varphi_1(\lambda, \mu)$ under a homology $\Omega(\mu)$ with center at O, axis the line l_λ, and parameter μ. This transform $\varphi_1(\lambda, \mu)$ forms a two-parameter family. The point M on $\varphi_1(\lambda, \mu)$ is now determined for given λ, μ so that its locus $S^{(1)}_{\varphi_1}$ is enveloped by the line MA as O generates C. There is a one-parameter family of such points M on $\varphi_1(\lambda, \mu)$. The cross ratios of four such points is a constant. Finally the tangent to $\varphi_1(\lambda, \mu)$ at M intersects the tangent to C at O in a point Q whose locus $S^{(2)}_{\varphi_1}$ is to be so determined that it is enveloped by QM. Again there is a one-parameter family of such points M (and Q). Four such envelopes $S^{(2)}_{\varphi_1}$ for the same conic $\varphi_1(\lambda, \mu)$ cut out points on any tangent to C with constant cross ratio.

The second paper varies these problems in the following manner. The osculating conic of C at O is replaced by the osculating conic K_λ to the branch of Γ_λ not tangent to C. The conics $\varphi_1(\lambda, \mu)$ are replaced by the transforms $\varphi_2(\lambda, \mu)$ of K_λ under the homology $\Omega(\mu)$ described above. The envelope $S^{(1)}_{\varphi_2}$ is defined as the locus as O varies on C of a point M on φ_2 whose tangent is the line MN (N being defined as the intersection of l_λ with the tangent to C). The envelope $S^{(2)}_{\varphi_2}$ is defined as follows. The line of inflexions l_λ is tangent to $\varphi_2(\lambda, \mu)$ for all (λ, μ). The envelope $S^{(2)}_{\varphi_2}$ is the locus of a point Q of l_λ whose tangent at Q is l_λ. Corresponding theorems concerning cross ratios are stated.

In the third paper the pencil φ_1 of conics of the first paper is replaced by the pencil $\varphi_3(\lambda, \mu)$ of transforms under $\Omega(\mu)$ of the conic enveloped by the join of conjugate points of the pencil Γ_λ of nodal cubics [cf. Salini, Atti Accad. Gioenia Catania (6) **2**, no. 4 (1937)]. The envelope $S^{(1)}_{\varphi_3}$ is the locus of points M (as O describes C) on any one of the conics $\varphi_3(\lambda, \mu)$ whose tangent at M is the line MP, P being defined as in the first paper. Let Q be the intersection of the tangent at M to any one of the conics $\varphi_3(\lambda, \mu)$ with the line PO. The envelope $S^{(2)}_{\varphi_3}$ is the locus of that point Q as O generates C, whose tangent at Q is the line QM. Some theorems on constancy of cross-ratios are again stated for these curves $S^{(1)}_{\varphi_3}$ and $S^{(2)}_{\varphi_3}$. *V. G. Grove* (East Lansing, Mich.).

Maeda, Jusaku. On the osculating crunodal circular cubic of a plane curve. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **30**, 287–318 (1942).

A study of particular cubic curves metrically associated with the neighborhood of a point of a plane curve.

E. Bompiani (Pittsburgh, Pa.).

Maeda, Kazuhiko. On certain isothermal systems in a plane. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **32**, 57–119 (1945).

In polar coordinates (r, θ), the curve $r^n = a^n \sin n\theta$ is a sine spiral of index n with pole at the origin. Denote by (n) the homothetic of the osculating circle to a curve (M) at a point M in the ratio $n:1$, with M for homothetic center. If $n^2 \neq 1$, there are an infinity of sine spirals of index n having contact of at least second order with (M) at M. The locus of poles of such spirals is a circle $((n+1)/2)$. The characteristic points of the circle $((n+1)/2)$ are the point M and the pole $P(n)$ of index n of (M) at M. The line joining M to the pole of index n is called the normal of index n of (M) at M. The curve all of whose normals of index n pass through a fixed point is called a Cesàro curve of index n.

The fixed point is termed the pole of the Cesàro curve. The envelope of normals of index n, that is, the locus of the poles of the osculating Cesàro curves, is called the evolute of index n. The second pole of index n is the pole of the osculating Cesàro curve of index n. The author solves completely the following problems: the determination of an isothermal net such that (1) the normal of index m and the normal of index n of the curves through any point coincide; (2) the pole of index m and the pole of index n of the two curves through any point coincide; (3) the second pole of index m and the second pole of index n of the two curves through any point coincide; (4) the net consists of congruent curves; (5) the net has special properties, e.g., if κ_1 and κ_2 are the curvatures of the two curves of the net through a given point, then $\kappa_1^2 \neq \kappa_2^2$ is constant, or κ_1/κ_2 is constant, or the homologic or inversive curvatures obey certain simple relations. *J. De Cicco* (Chicago, Ill.).

Maeda, Kazuhiko. On certain circular cubic surfaces associated with a tangent to a surface. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **32**, 149–164 (1945).

A study of metric properties of the plane sections of a surface at a given point and with a given tangent. Evolutes of these sections and particular points attached to them are also examined. To give an idea of the kind of theorems obtained, it suffices to give this: the locus of the foci of conics having a contact of the third order (at least) with a surface at a given point and direction is a cubic circular surface. *E. Bompiani* (Pittsburgh, Pa.).

Correnti, Salvatore. Sulle forme tipiche del ds^2 delle superficie a curvatura costante. Matematiche, Catania **3**, 40–48 (1948).

The author computes ds^2 for a surface of constant curvature referred to minimal parameters. *V. Hlavatý.*

Lemoine, Simone. Sur les surfaces admettant deux formes linéaires données comme éléments d'arc de leurs lignes de courbure. C. R. Acad. Sci. Paris **228**, 461–463 (1948).

In a surface let K be the total curvature, and ω_1 and ω_2 two independent linear differential forms. Define r and s by the relations: $d\omega_1 = r[\omega_1\omega_2]$; $d\omega_2 = s[\omega_2\omega_1]$; and let $\xi = (s/K)_2 + rs/K$ and $\eta = (r/K)_1 + rs/K$, where the subscripts denote differentiation relative to ω_1 and ω_2. It is proved that if $K \neq 0$ and at least one of ξ and η are not zero, then there exist at most two essentially distinct surfaces having ω_1 and ω_2 as elements of arc of their lines of curvature. This result is included in a theorem of É. Cartan [Les systèmes différentiels extérieurs et leurs applications géométriques, Actual. Sci. Ind., no. 994, Hermann, Paris, 1945, p. 154].

C. B. Allendoerfer (Haverford, Pa.).

Lalan, Victor. Le rôle du tenseur moyen dans la détermination des directions principales. C. R. Acad. Sci. Paris **228**, 536–538 (1949).

A new proof is given of the following local theorem of H. W. Alexander [Trans. Amer. Math. Soc. **47**, 230–253 (1940); these Rev. **1**, 269]. If a surface contains no umbilical points and if the lines of curvature are not isometric, then the second fundamental tensor may be expressed algebraically in terms of the first fundamental tensor and the mean curvature. The proof makes use of the Cartan calculus of differential forms, whereas Alexander uses tensor calculus. Lalan's proof is simpler, but Alexander's final result is more explicit. *C. B. Allendoerfer* (Haverford, Pa.).

Pinl, M. Über Flächen mit isotropem mittleren Krümmungsvektor. Monatsh. Math. **52**, 301–310 (1948).

Let R_n be an n-dimensional complex Euclidean space. Consider a two-dimensional surface imbedded in R_n whose parameters are isotropic but whose metric tensor is not zero. The author investigates those surfaces whose mean curvature vector is isotropic. He shows that, if $n=3$, there are no surfaces having isotropic mean curvature. If $n=4$, many such surfaces exist. The discussion divides according as the rank r of a certain matrix is 1, 2 or 3. The general case, $r=3$, is discussed in detail and the solution of the corresponding Monge equation is given without indicated quadratures. *A. Fialkow* (Brooklyn, N. Y.).

Longo, C. Sopra una classe di varietà che ammettono varietà subordinate quasi-asintotiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 19–21 (1948).

Une sous-variété V_h d'une variété V_k est quasi-asymptotique (r, s) si l'espace r-osculateur à V_k en un point de V_h et l'espace s-osculateur en ce point à V_h ont une intersection de dimension plus grande que pour une V_h générique de V_k. L'auteur construit ici le premier exemple de V_{k+1} ayant des V_h quasi-asymptotiques pour toutes les valeurs de h $(2 \leqq h \leqq k)$ en généralisant un travail antérieur de Bogdan. On considère la Veronesienne Φ qui représente toutes les formes d'ordre n de S_k. L'espace $(n-1)$-osculateur en un point variable P de Φ coupe l'espace tangent en un point fixe A de Φ en Q: V_{k+1} est le lieu des droites PQ. Si l'on désigne le point courant de V_{k+1} par $M = P + \rho Q$ les variétés V_h quasi-asymptotiques sont définies par $\rho = f(x_1, \cdots, x_h)$, $x_{h+i} = \varphi_i(x_1, \cdots, x_h)$, $x_1, \cdots, x_k$ étant les coordonnées cartésiennes dans S_k et f une fonction homogène de degré $n-1$, les φ_i des fonctions homogènes de degré 1.

L. Gauthier (Nancy).

de Donder, Théophile, et van den Dungen, Frans-H. Sur le mouvement relatif des corps solides. C. R. Acad. Sci. Paris **228**, 221–223 (1949).

Soit O, X^a $(a=1, 2, 3)$ un trièdre initial; le mouvement d'un solide porteur d'un trièdre G, ξ^a est donné par $X^a = H_b{}^a\xi^b + X_G{}^a$. Le mouvement est régi par les équations aux dérivées variationelles $\delta L/\delta X_G{}^a = 0$, $\delta L/\delta H_b{}^a = 0$, où L est la fonction lagrangienne. Supposons maintenant qu'un trièdre o, x^a se meut par rapport au trièdre initial dans un mouvement connu, on aura $x^a = h_b{}^a\xi^b + x_G{}^a$. Les auteurs démontrent qu'on a pour le mouvement relatif $\delta L^*/\delta x_G{}^a = 0$, $\delta L^*/\delta h_b{}^a$, où L^* est la fonction lagrangienne du mouvement absolu calculée en fonction des variables du mouvement relatif. *O. Bottema* (Delft).

Hulubei, Dan. Déplacements dans un espace euclidien à 4 dimensions. Disquisit. Math. Phys. **6**, 129–136 (1948).

An infinitesimal motion in an n-dimensional Euclidean space E_n defines a null polarity for which every point P corresponds to the E_{n-1} through P orthogonal to the velocity vector of P. Conversely a null polarity defines an infinitesimal motion. For even n a null polarity is singular and the E_{n-1}'s of this polarity all pass through a fixed point O. The section of the polarity with an E_{n-1} not passing through O is a nonsingular null polarity in this E_{n-1} and therefore induces a motion in E_{n-1}. The author shows that the results obtained by Woinaroski in the first part of his paper [Disquisit. Math. Phys. **4**, 175–239 (1945); these Rev. **8**, 532] can all be immediately obtained from the properties of the polarity mentioned above. *J. Haantjes* (Leiden).

Vincensini, Paul. Sur la géométrie des champs de vecteurs unitaires. C. R. Acad. Sci. Paris **227**, 952–954 (1948).

Soit $v^h(x)$ un champ de vecteurs unitaires dans un espace cartésien, (C) la congruence du champ et (D) celle de courbes le long desquelles v^h a une direction fixe. On peut associer à chaque courbe D l'une des normales N d'une surface S de la même direction v^h. Soit $x \leftrightarrow \mu$ une correspondance entre les points x et μ de D et N. La transformation $v^h(x) \rightarrow v^h(x')$, $x' = x + O\mu$, conserve la circulation sur tout contour fermé [mêmes C. R. **226**, 1163–1165 (1948); ces Rev. **9**, 615]. La transformation conserve en même temps la congruence (C) lorsque l'indicatrice des tangents de C est l'image sphérique d'une ligne de courbure de S et μ est un centre principal sur N. Il résulte que les congruences (C) et (D) ont ∞^1 surfaces communes et sur ces surfaces les courbes D constituent une famille de lignes d'ombre admettant les C pour conjuguées. *J. Haantjes* (Leiden).

Mayer, O. Remarques sur la décomposition en homologies harmoniques d'une collinéation hermitienne. Ann. Sci. Univ. Jassy. Sect. I. **30** (1944–1947), 37–42 (1948).

E. Bertini, précisant un résultat de Voss, a énoncé la proposition suivante. Si une collinéation C conservant une quadrique Q non dégénérée de l'espace projèctif S_r possède un espace principal S_m, elle est le produit de $r-m$ homologies conservant la quadrique Q. L'auteur montre que cette proposition présente une exception, du fait que, pour $r-m$ pair, Q peut admettre des collinéations biaxiales paraboliques, dont le premier axe S_m (qui est espace principal) touche Q suivant le deuxième axe S_n $(n=r-m-1)$. Il établit que, C étant une collinéation qui conserve la quadrique non dégénérée Q de S_r possédant un S_m principal, si C n'est pas biaxiale parabolique elle est décomposable en $r-m$ homologies (dont le nombre ne peut être abaissé) conservant Q dont les hyperplans sont indépendants et passent par S_m, la décomposition exigeant $r-m+2$ homologies dans le cas où C est biaxiale parabolique. Comme applications de ce résultat, l'auteur retrouve et améliore le théorème initial de Voss, et signale une démonstration de cet autre théorème de Voss: dans un S_r de dimension impaire, les espaces principaux de première espèce (s'il en existe) sont de dimension impaire, ceux de deuxième espèce étant de dimension paire. *P. Vincensini* (Besançon).

Mayer, O. Sur la composition du groupe de collinéations d'une quadrique non-dégénérée dans un espace projectif complexe. Ann. Sci. Univ. Jassy. Sect. I. **30** (1944–1947), 229–233 (1948).

Dans cette note l'auteur établit, par des considérations projectives élémentaires utilisées dans la note précédemment analysée, le théorème suivant: dans un S_r projectif complexe, le groupe des collinéations conservant une quadrique non dégénérée est simple si r est pair; si r est impair et supérieur à 3 ce groupe contient un seul sous-groupe invariant propre formé par les collinéations (de première espèce) transformant en eux-mêmes les deux systèmes d'espaces maximaux de la quadrique, et ce sous-groupe est simple. La démonstration s'appuie sur le fait que le groupe en question peut transformer l'un dans l'autre deux S_m (non tangents) donnés, et sur le théorème de Voss dont il est question dans la note précédente, suivant lequel toute collinéation du groupe conservant une quadrique non dégénérée peut être décomposée en homologies harmoniques du même groupe. *P. Vincensini* (Besançon).

Norden, A. P. The Riemannian metric on surfaces of a projective space. Doklady Akad. Nauk SSSR (N.S.) **60**, 345–347 (1948). (Russian)

In earlier papers [Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] **5**, 226–245 (1941); these Rev. **8**, 604, and one said to have appeared in the same collection, **6** (1948)] the author has shown how, given a net on a surface in projective space, one can define a pair of metrics of Weyl. In this paper the case is considered when these metrics are Riemannian. Such nets are called J nets and they can be characterized by the fact that their axes and edges of Green form congruences which are respectively conjugate and harmonic to the surface. Four special types of J nets are defined as follows. A net is a J_1 net if the tangents to the curves of both of its families are tangent to the same surface of degree two or of class two which does not degenerate into two planes and does not coincide with the given surface; on every surface there exist ∞^9 J_1 nets. A net is a J_0 net if it consists of two families of plane curves whose planes belong to two bundles whose axes intersect; on every surface there exist ∞^7 J_0 nets. A net is a J_1' net if it is neither J_1 nor J_0 and if it determines two Riemannian metrics with nonzero curvatures; if these geometries are Euclidean the net is called a J_0' net. The author discusses the relation of these nets to those of Jonas, Koenigs, and those which Dubnov calls nets of Ricci. In discussing projective deformations the author uses for projective applicability a definition in terms of internal geometries (equivalent to the definition of Fubini). Under deformation a J net is transformed into a J net. If the surface is deformable there are ∞^{9+k} J_1 nets and ∞^{7+k} J_0 nets, where k is the number of parameters of deformation. The existence of J_0' nets characterizes deformable surfaces. *G. Y. Rainich.*

Bušmanova, G. V., and Norden, A. P. Projective invariants of a normalized surface. Doklady Akad. Nauk SSSR (N.S.) **60**, 1309–1312 (1948). (Russian)

A normalized surface in three dimensional projective space is given in point and in tangential coordinates by $x^\alpha = x^\alpha(u^i)$; $\xi_\alpha = \xi_\alpha(u^i)$, $\alpha = 1, \cdots, 4$; $i = 1, 2$, respectively. Its normal of the first kind is determined by the intersection of the plane $\eta_{\alpha i} = \partial_i \xi_\alpha - \lambda_i \xi_\alpha$, and its normal of the second kind by the combination with the point $y_i^\alpha = \partial_i x^\alpha - l_i x^\alpha$. The coefficients of its interior connections of the first and second kind are connected by a relation containing the tensor $\bar{b}^{im} \nabla_k b_{jm} - \omega_k \delta_j^i$, where b_{ij} is the tensor of the asymptotic net, b^{ij} its reciprocal, the differentiation is taken with respect to the coefficients of the connection of the first kind, the tensor $\omega_k = \bar{b}^{im} \nabla_k b_{im} - 4T_i$, and the tensor $T_i = \frac{1}{2} \bar{b}^{mn} (\nabla_m b_{ni} - \frac{1}{2} \nabla_i b_{mn})$ is the Chebyshev tensor of the first kind of the asymptotic net.

A number of relations follow concerning the tensor a_{ij}^k of Fubini-Čech, the projective normal of the surface, the invariant $J = -\frac{1}{8} \bar{b}^{mn} a_{ml}^k a_{kn}^l$ of Fubini-Čech, the directrix of Wilczynski and other quantities. The case is discussed where the normal of the first kind coincides with the metrical normal of a surface in Euclidean space, and the normal of the second kind lies in the plane at infinity. The Gauss-Codazzi equations are $\omega_i = 0$, $T_i = \frac{1}{4} \partial_i \ln K$, where K is the Gaussian curvature. The tensors a_{jk}^i and J are expressed for this case, as well as $\nabla_k b_{ij}$. The paper ends with the equation of the surface of revolution of which the metrical normal

coincides with the projective normal:

$$x=u\cos v,\quad y=u\sin v,\quad z=z(u),$$
$$(1+z'^2)^4(uz'z'''-3uz''^2+3z'z'')^3=Cu^7(z')^9(z'')^{12}.$$

The article refers to papers by Norden [Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] 6 (1948) (unavailable to the reviewer)], Efimov [ibid. 5, 148–172 (1941); these Rev. 8, 346], Kovancov [same Doklady (N.S.) 58, 1261–1263 (1947); these Rev. 9, 377], and Čahtauri [same Doklady (N.S.) 59, 1257–1259 (1948); these Rev. 9, 531]. *D. J. Struik.*

Sen, R. N. Parallel displacement and scalar product of vectors. Proc. Nat. Inst. Sci. India 14, 45–52 (1948).

The author considers a parallel displacement of a vector in a Riemannian space, defined by means of arbitrary coefficients of connection. It is shown that this parallelism gives rise to an associated parallelism. The scalar product of two vectors remains unchanged when one vector is given a parallel transport and the other is given its associate transport. Some properties involving the two kinds of parallel displacement of vectors around an infinitesimal closed circuit are deduced. The general form of a parallelism based upon symmetric coefficients of connection is established. The parallelisms of Weyl and of Levi-Civita are discussed. *A. Fialkow* (Brooklyn, N. Y.).

Sorace, Orazio. Trasporti rigidi di vettori. Matematiche, Catania 3, 59–67 (1948).

I. Let $L(\xi^\nu=\xi^\nu(t))$ be a given curve in a V_3 referred to the coordinate system ξ^ν, $\Gamma^\nu_{\lambda\mu}$ the Christoffel symbols belonging to the metric tensor, $F^{\nu\lambda}\equiv F^{[\nu\lambda]}$ a given skew symmetric tensor, $\Lambda^\nu_\lambda\equiv\Gamma^\nu_{\lambda\mu}\xi^{\mu\prime}+F^{\cdot\nu}_{\lambda}$ and δ the symbol of covariant differential (along L) with respect to $\Lambda_\lambda{}^\nu$. Then

$$\text{(1)}\qquad \frac{\delta}{dt}\underset{a}{i^\nu}=-\underset{a-1}{\kappa}\,\underset{a-1}{i^\nu}+\underset{a}{\kappa}\,\underset{a+1}{i^\nu},\qquad a=1,2,3,\ \underset{0}{\kappa}=\underset{3}{\kappa}=0,$$

where $\underset{1}{i^\nu}$, $\underset{2}{i^\nu}$, $\underset{3}{i^\nu}$ are, respectively, the unit tangential vector of L and the first and second normal unit vectors of L with respect to δ, and $\underset{1}{\kappa}$, $\underset{2}{\kappa}$ are its curvatures (with respect to δ). If $R^\nu=\overset{a}{R}\underset{a}{i^\nu}$ is a vector field of pseudoparallel vectors, $(\delta/dt)R^\nu=0$, then one gets easily from (1)

$$\text{(2)}\qquad \overset{1}{R}'=\overset{1}{K}\overset{2}{R},\quad \overset{2}{R}'=-\overset{1}{K}\overset{1}{R}-\overset{2}{K}\overset{3}{R},\quad \overset{3}{R}'=\overset{2}{K}\overset{2}{R}$$
$$(\overset{1}{K}\equiv\underset{1}{\kappa};\ \overset{2}{K}=-\underset{2}{\kappa}).$$

II. Let

$$\overset{2}{R}(t)=\lambda(t)+\int_{t_0}^t Q(t,s)\lambda(s)ds$$

be the solution of a Volterra equation

$$\overset{2}{R}(t)=\lambda(t)-\int_{t_0}^t N(t,s)\overset{2}{R}(s)ds,$$

where

$$N(t,s)=\sum_{i=1}^{2}\overset{i}{K}(s)\int_s^t\overset{i}{K}(x)dx,$$

$$\lambda(t)=\overset{2}{\underset{0}{R}}-\overset{1}{\underset{0}{R}}\int_{t_0}^t\overset{1}{K}(x)dx-\overset{3}{\underset{0}{R}}\int_{t_0}^t\overset{2}{K}(x)dx$$

$$(\overset{a}{\underset{0}{R}}=\text{constant},\ a=1,2,3),$$

and put

$$\overset{1}{R}(t)=\overset{1}{\underset{0}{R}}+\int_{t_0}^t\left[\overset{1}{K}(s)+\int_s^t Q(x,s)\overset{1}{K}(x)dx\right]\lambda(s)ds,$$

$$\overset{3}{R}(t)=\overset{3}{\underset{0}{R}}+\int_{t_0}^t\left[\overset{2}{K}(s)+\int_s^t Q(x,s)\overset{2}{K}(x)dx\right]\lambda(s)ds.$$

The functions $R(t)$, $a=1, 2, 3$, satisfy (2) and consequently represent the solution of (2). This is the main result of the paper, presented in a slightly different way from that used by the author. *V. Hlavatý.*

Wrona, Włodzimierz. On multivectors in a V_n. I. Nederl. Akad. Wetensch., Proc. 51, 1291–1301 = Indagationes Math. 10, 435–445 (1948).

Dans une variété riemannienne V_n, de tenseur fondamental $a_{\lambda\mu}$ et de tenseur de courbure $R_{\kappa\lambda\mu\nu}$, l'auteur introduit à partir du tenseur $U_{\kappa\lambda\mu\nu}=R_{\kappa\lambda\mu\nu}+2Ka_{[\kappa[\mu}a_{\lambda]\nu]}$, où K désigne la courbure riemannienne scalaire de V_n, l'ensemble des tenseurs

$$\overset{m}{U}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m}=\frac{m!}{2}U_{[\lambda_1\lambda_2[\mu_1\mu_2}a_{\lambda_3\mu_3}\cdots a_{\lambda_m]\mu_m]},$$

où $2\leqq m\leqq n$ et où les notations sont celles de Schouten. A partir des tenseurs $\overset{m}{U}$, il donne des formes nouvelles des conditions pour que l'espace soit à courbure constante ou espace d'Einstein ou conforme à un espace euclidien, ainsi qu'une expression simple de la courbure scalaire dans une m-direction. L'extension classique du théorème de Schur apparaît comme une conséquence presque triviale de cette expression de la courbure scalaire. Certains résultats coïncident avec ceux publiés ailleurs par Haantjes et l'auteur. *A. Lichnerowicz* (Strasbourg).

Castoldi, L. Attorno a un "teorema della divergenza" per tensori qualunque negli spazi di Riemann. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 395–398 (1948).

The author generalises the well-known integral theorem of Green by expressing the integral of the divergence of a tensor of any order, taken over a region of Riemannian space V_n, as a surface integral, taken over the $(n-1)$-dimensional boundary of the given region. The second integral is interpreted in the usual manner as the flux of the tensor across the boundary in question. *A. J. McConnell* (Dublin).

Castoldi, L. Applicazioni dei teoremi generalizzati della divergenza e di Stockes al calcolo delle derivate sostanziali di integrali multipli negli spazi di Riemann. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 398–402 (1948).

Both the integral theorems of Green and Stokes can be extended to apply to an m-dimensional subspace V_m, immersed in a Riemannian space V_n. The author considers a fluid occupying the region V_m and moving in a prescribed manner through the surrounding space V_n. By using these extensions of Green's and Stokes's theorems, he obtains rather complicated tensor formulae for the rate of change, following the fluid, of the "normal flux" and the "tangential flux" of a tensor of any order, taken over the region of the fluid. *A. J. McConnell* (Dublin).

***Finzi, Bruno, e Pastori, Maria. Calcolo Tensoriale e Applicazioni.** Nicola Zanichelli, Bologna, 1949. vii+427 pp.

The book is divided into three different parts. The first part consists of five introductory chapters, the second part contains two chapters on geometrical applications, and the last part deals (in three chapters) with physical applications. The method used is chiefly an intuitive one; the topics are standard results. A reader interested in the historical development of the Italian school of "vector calculus" will find many significant remnants of it in the first part. The achievements of the recent Italian school [Bompiani, Bortolotti, Maxia, etc.] are not dealt with.

The first chapter deals with introductory statements about the physical significance of various classical symbols used in the vector calculus of three-dimensional Euclidean space E_3 (div, rot, grad, Δ, etc.). The second chapter is devoted to an intuitive introduction to the algebra of tensor calculus of E_3, more or less under the influence of the classical Italian school. This influence is stressed in the third chapter (on quadratic tensors in E_3), which recalls the "tempi passati" of Burali-Forti and Marcolongo. In the next chapter the reader finds the beginning of tensor analysis in E_3. Finally, the usual tensor analysis for a Riemann space V_m in E_n is developed in the fifth chapter, which ends with some remarks about some more general connections [Weyl, etc.]. In the sixth and seventh chapters the foregoing results are applied to the theory of a Riemann space V_n ($n=2$, $n>2$) based (for $n=2$) on both fundamental tensors. In particular, the Ricci congruence calculus ("the coefficients of Ricci") is used and the properties of the curvature tensor are derived. The physical applications are contained in three last chapters: theory of deformations [chapter VIII], the electromagnetic field [chapter IX] and finally the theory of relativity [chapter X]. *V. Hlavatý.*

Kosambi, D. D. The differential invariants of a two-index tensor. Bull. Amer. Math. Soc. **55**, 90–94 (1949).

Given a tensor $g_{ij}(x)$ with the nonvanishing determinant of its symmetrical part $g_{(ij)}$, introduce two parameters u, v and consider the variational problem

$$\delta\int g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}du\,dv=0.$$

Its Euler equation leads to the connection coefficients

$$\Gamma^i_{jk}=\{{}^{\,i}_{jk}\}+T^i_{jk},$$

where $\{{}^{\,i}_{jk}\}$ are the Christoffel symbols with respect to $g_{(ij)}$ and the tensor T^i_{jk} is defined as follows:

$$T^r_{jk}=\tfrac{1}{2}\left(\frac{\partial}{\partial x^j}g_{[ik]}+\frac{\partial}{\partial x^k}g_{[ji]}+\frac{\partial}{\partial x^i}g_{[kj]}\right)g^{ri}.$$

Introducing a single parameter t and considering the variational problem

$$\delta\int\left\{g_{(ij)}\dot{x}^i\dot{x}^j+\left(-\frac{\partial}{\partial x^k}g_{ij}/2+g_{ir}A^r_{jk}\right)\dot{x}^i\dot{x}^j\dot{x}^k\right\}dt=0$$

with properly chosen A and det $|g_{(ij)}|\neq 0$ one gets by means of the corresponding Euler equation the connection coefficients

$$\gamma^r_{(jk)}=g^{ir}(g_{ij,k}+g_{ik,j}-H_{ijk})/3$$

with

$$H_{ijk}=g_{ir}\{{}^{\,r}_{jk}\}+g_{jr}\{{}^{\,r}_{ki}\}+g_{kr}\{{}^{\,r}_{ij}\}$$

from which again a connection L^r_{jk} may be built up.
V. Hlavatý (Bloomington, Ind.).

Levine, Jack. Fields of parallel vectors in projectively flat spaces. Duke Math. J. **16**, 23–32 (1949).

The main features may be described as follows. The integrability condition $R_{\omega\mu\lambda}{}^{\nu}\underset{a}{v}{}^{\lambda}=0$ for the parallel displaced vector fields in a projectively flat space with the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ ($a=1, \cdots, p$), show that $R_{\omega\mu\lambda}{}^{\omega}$ is symmetric. Hence the coefficients of such a connection may be supposed in the form

$$(1)\qquad \Gamma^{\nu}_{\lambda\mu}=-(\delta^{\nu}_{\lambda}q_{\mu}+\delta^{\nu}_{\mu}q_{\lambda}),\quad q_{\lambda}=\frac{\partial}{\partial x^{\lambda}}q,$$

and the equations for parallel displacement mentioned above are

$$\frac{\partial}{\partial x^{\mu}}\underset{a}{v}{}^{\nu}=q_{\mu}\underset{a}{v}{}^{\nu}+\delta_{\mu}{}^{\nu}(\underset{a}{v}{}^{\alpha}q_{\alpha})$$

with the solution

$$\underset{a}{v}{}^{\nu}=(\underset{a}{c}x^{\nu}+\underset{a}{d}{}^{\nu})e^{q},\qquad c, d=\text{constant},$$

where

$$(\underset{a}{c}x^{\lambda}+\underset{a}{d}{}^{\lambda})\frac{\partial}{\partial x^{\lambda}}e^{q}=\underset{a}{c}e^{q}.$$

The author solves the last equation explicitly for q and gets in this way the most general connection (1) which admits p parallel contravariant vector fields. A similar treatment of parallel covariant vector fields ω_{λ} leads to $p=1$ (for a nonflat connection). There also the author gives the explicit solution for ω_{λ} and q. *V. Hlavatý* (Bloomington, Ind.).

Yen, Chih Ta. Sur l'équivalence des formes différentielles extérieures quadratiques à quatre variables. C. R. Acad. Sci. Paris **227**, 817–819 (1948).

Sur une variété V_{2n}, n fois différentiable, soit Ω une forme différentielle quadratique extérieure de rang $2n$ et $n-1$ fois différentiable. Si Ω est complètement intégrable, il existe une forme de Pfaff bien déterminée telle que $d\Omega=\theta\wedge\Omega$. Ehresmann et Libermann ont examiné [mêmes C. R. **227**, 420–421 (1948); ces Rev. **10**, 122] le cas où $n>2$; alors $d\theta=0$ et les formes Ω complètement intégrables sont réductibles à des formes canoniques et admettent un groupe infini de transformations ponctuelles. Le présent papier est relatif au cas $n=2$. Une forme Ω de rang 4 et 2 fois différentiable est alors toujours complètement intégrable. Si $\theta\wedge d\theta=0$, Ω admet un groupe infini de transformations ponctuelles. L'auteur se place ici dans le cas "général" où $d\theta$ est une forme quadratique non dégénérée; on peut alors ramener Ω et $d\theta$ à la forme $\Omega=\omega^1\wedge\omega^2+\omega^3\wedge\omega^4$; $d\theta=\lambda[\omega^1\wedge\omega^2-\omega^3\wedge\omega^4]$. On suppose de plus $\theta=\omega^1+\omega^3$, $d\lambda=\lambda_2\omega^2+\lambda_4\omega^4$. Les 4 formes de Pfaff ω^i sont alors déterminées d'une manière unique et le problème d'équivalence de Ω est ramené au problème d'équivalence spécial d'un système de 4 formes de Pfaff. Il existe 12 invariants fondamentaux. Le problème d'équivalence correspondant pour une équation différentielle extérieur $\Omega=0$ est aussi analysé. *A. Lichnerowicz* (Strasbourg).

Yen, Chih-Ta. Sur l'équivalence des formes différentielles extérieures quadratiques à quatre variables. C. R. Acad. Sci. Paris **227**, 1203–1204 (1948).

Cette seconde note [cf. l'analyse précédente] est consacrée au cas où $d\theta$ est dégénérée (avec $\theta\wedge d\theta\neq 0$). Les

formes Ω, θ, $d\theta$ peuveut être réduites aux formes normales $\Omega=\omega^1\wedge\omega^2+\omega^3\wedge\omega^4$; $\theta=\omega^2$; $d\theta=\omega^1\wedge\omega^3$, où ω^2 et ω^3 sont des formes de Pfaff intrinsèquement définies. Les principaux cas à étudier sont celui où le champ $\omega^1=\omega^3=0$ n'est pas complètement intégrable et celui où ce champ est complètement intégrable, le champ $\omega^3=0$ ne l'étant pas. Dans le premier cas, il existe 8 invariants fondamentaux et Ω peut admettre un groupe di Lie à 4 paramétres. Dans le second, il existe 7 invariants fondamentaux et Ω ne peut admettre aucun groupe à 4 paramètres. Le résultat principal de l'ensemble de ces deux notes est le suivant: lorsque le groupe qui laisse invariant une forme Ω de rang 4 à 4 variables est un groupe de Lie, ce groupe admet au plus 4 paramètres.

A. Lichnerowicz (Strasbourg).

Haimovici, M. Sur la géométrie des familles de transformations ponctuelles simplement transitives. Ann. Sci. Univ. Jassy. Sect. I. **30** (1944–1947), 1–36 (1948).

Dans des travaux classiques [Cartan, J. Math. Pures Appl. (9) **6**, 1–119 (1927)], É. Cartan et Schouten ont montré qu'on peut associer à une variété de groupe à n paramètres deux connexions affines sans courbure. L'auteur considère ici une famille simplement transitive de transformations d'un espace à n dimensions V_n en lui même: (1) $X^i=F^i(x^j, a^k)$ $(i, j, k=1, 2, \cdots, n)$ qui en général ne définit pas un groupe. Il définit sur l'espace à $2n$ dimensions $V_n\times V_n$ une connexion affine intrinsèquement attachée à la famille (1), la méthode employée étant voisine de celle de Cartan. Si les transformations (1) forment un groupe, la courbure associée à la connexion est nulle et réciproquement. Lorsqu'il en est ainsi, la connexion affine introduite définit sur V_n une connexion affine à courbure nulle, pour laquelle les composantes de la torsion peuvent être choisies constantes. Les équations (1) définissant le premier groupe des paramètres d'un groupe donné, la connexion affine de l'auteur coïncide alors avec la connexion de seconde espèce de Cartan.

A. Lichnerowicz (Strasbourg).

Haimovici, Mendel. La géométrie des familles de transformations de variables dépendant de paramètres. Disquisit. Math. Phys. **6**, 81–128 (1948).

Le problème traité ici par l'auteur est une généralisation de celui traité dans un papier antérieur [cf. référat précédent]. L'auteur considère maintenant une famille transitive de transformations d'un espace à n dimensions V_n en lui-même: (1) $X^i=F^i(x^j, a^\alpha)$ $(i, j=1, \cdots, n; \alpha=1, \cdots, N\geqq n)$. Si l'on considère les X^i comme des constantes d'intégration, on peut associer à (1) un système de Pfaff de la forme (2) $\bar\omega^i(a, x, dx)=\omega^i(a, x, da)$, les ω^i et $\bar\omega^i$ étant des formes de Pfaff indépendantes. Dans l'espace à N dimensions des paramètres a, on considère les éléments plans à $N-n$ dimensions définis par les équations $\omega^i=0$. Si $n<N$, pour des valeurs données des x^i, ces équations définissent en chaque point un élément plan. Ce sont les éléments plans qui sont les éléments générateurs de l'espace à connexion affine T attaché par l'auteur à la famille (1). La connexion affine se trouve définie par la formulation de cinq axiomes de nature intrinsèque. Le papier comporte une étude détaillée de la courbure et de la torsion correspondante, ainsi que l'étude du cas particulier où les transformations (1) forment un groupe.

A. Lichnerowicz (Strasbourg).

Sasaki, Shigeo. On conformal normal coordinates. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **30**, 71–80 (1941).

Consider a space having a normal conformal connection. It is proved that the coordinate transformations among conformal normal coordinate systems with the same (arbitrary) point of the space are the set of enlarged conformal transformations if and only if the space is flat. This theorem contradicts an erroneous result of V. A. Hoyle [Princeton thesis, 1930]. The source of Hoyle's error is indicated.

A. Fialkow (Brooklyn, N. Y.).

***Yano, Kentaro. Groups of Transformations in Generalized Spaces.** Akademeia Press Company Ltd., Tokyo, 1949. iv+70 pp.

Die Ausdehnung des klassischen Bewegungsbegriffes der euklidischen Geometrie auf die allgemeinen Räume der Differentialgeometrie geschieht mit Hilfe der Theorie der kontinuierlichen Transformationsgruppen. Vorliegendes Buch gibt einen zusammenfassenden Überblick über die diesbezüglichen Untersuchungen. Bekanntlich kann die durch eine Transformationsgruppe bestimmte Geometrie eines Raumes auch durch ein System von integrablen Differentialformen bestimmt werden. In den neueren differentialgeometrischen Untersuchungen ist gerade der nicht-integrable Fall von Wichtigkeit. Die Formen bestimmen dann die lineare Übertragung (Zusammenhang) und damit, wie wohl bekannt, die Raumstruktur. Dementsprechend werden affinzusammenhängende, Riemannsche (metrische), konforme und projektivzusammenhängende Räume behandelt. Diejenigen Transformationen dieser Räume in sich, die den Zusammenhang invariant lassen, bilden eine Gruppe, die für die obigen Räume der Reihe nach als affine Transformationen, Bewegungen, konforme Transformationen und Kollineationen bezeichnet werden. Methodisch bedient sich Verf. der sogenannten Lieschen "Ableitungen" [Lie derivatives]. Diese bilden die Verallgemeinerung der Lieschen Operatoren auf beliebige geometrische Objekte. Die daraufbezüglichen Auseinandersetzungen werden in Kapitel 1–2 entwickelt, während Kapitel 3–6 die oben erwähnten Transformationen zum Gegenstand haben. Verf. kann unter konsequenter Verwendung der Lieschen Ableitungen ausser einer Reihe von ihm herrührenden neuen Untersuchungen auch die schon bekannten Ergebnisse anderer Autoren in sehr eleganter und übersichtlicher Weise herleiten. Dabei werden auch bei Betrachtungen von mehr formaler Natur überall geometrische Deutungen gegeben. Dieses Buch wird für alldiejenigen, die sich mit diesem Gegenstand beschäftigen, ein guter Wegweiser sein.

O. Varga (Debrecen).

Freeman, J. G. A generalization of minimal varieties. Proc. Edinburgh Math. Soc. (2) **8**, 66–72 (1948).

The author considers a geometry based on a fundamental function $L(x, u)$, where x are the space coordinates in S_n and u the components of a contravariant vector-density of weight p, the "element of support" [cf. J. A. Schouten and J. Haantjes, Monatsh. Math. Phys. **43**, 161–176 (1936); J. G. Freeman, Quart. J. Math., Oxford Ser. **15**, 70–83 (1944); these Rev. **6**, 188]. A space S_r in S_n is called a minimal variety if for any region R in S_r bounded by a given closed hypersurface B_{r-1} the first variation of the volume integral vanishes for arbitrary displacements of points and of the element of support, which are zero at the boundary. The conditions for a minimal variety S_r in S_n are given in tensor form. As a special case it is shown that a Finsler space $(p=0)$ can possess a minimal variety S_2 with tangential element of support only in restricted cases. Furthermore the author defines a mean curvature of a subspace S_r with respect to a given normal and it is proved that if S_r is a minimal variety to which the element of support is normal,

its mean curvature with respect to the element of support vanishes. *J. Haantjes* (Leiden).

Iwamoto, Hideyuki. On geometries associated with multiple integrals. Math. Japonicae 1, 74–91 (1948).

Supposons donnée, dans une variété différentiable V_n, une intégrale k-multiple

$$O=\int L(x, p)du^1 \cdots du^k,$$

l'intégrale étant étendue à une variété à k dimensions plongée dans V_n et L dépendant localement des x^i, des $p_\lambda{}^i=\partial x^i/\partial u^\lambda$ et satisfaisant à la condition d'invariance $p_i^\lambda p_\mu{}^i=\delta_\mu{}^\lambda$ ($p_i^\lambda=\partial \log L/\partial p_\lambda{}^i$). La plus grande partie de ce papier est consacrée au problème suivant, déjà étudie par Davies et d'autres auteurs: construire des géométries associées d'une manière invariante à l'intégrale O. Pour $k=1$, on a le problème fondamental de la géométrie des espaces de Finsler et pour $k=n-1$ celui des espaces de Cartan. La forme de Legendre de L est définie par

$$L_{ij}^{\lambda\mu}=\frac{1}{L}\frac{\partial^2 L}{\partial p_\lambda{}^i\partial p_\mu{}^j}-p_i^\lambda p_j^\mu+p_i^\mu p_j^\lambda.$$

Dans le chapitre 1, l'auteur étudie le cas où cette forme est décomposé, c'est-à-dire où il existe des quantités $g_{ij}^{\prime\prime}$, $g^{\lambda\mu}$ telles que $L_{ij}^{\lambda\mu}=g_{ij}^{\prime\prime}g^{\lambda\mu}$. Il est alors possible de déterminer algébriquement d'une manière unique un tenseur fondamental g_{ij} et une connexion euclidienne associée; il en est ainsi dans les cas de Finsler et Cartan. Dans le cas général étudié au chapitre 2, l'auteur indique une méthode intégrale nouvelle permettant d'attacher d'une manière intrinsèque à l'intégral O une connexion euclidienne, qui se réduit aux connexions usuelles dans les cas précédemment étudiés. Cette méthode présentant l'inconvénient d'être transcendante, l'auteur étudie au chapitre 3 la détermination rationnelle, sur l'espace fibré F des éléments plans (x, p), d'une connexion affine attachée à O; il utilise à cet effet un nouvel espace fibré sur F comme base. Le papier se termine par l'esquisse d'une généralisation à des intégrales où figurent des dérivées d'ordre supérieur. *A. Lichnerowicz.*

Alardin, Félix. L'autoparallélisme des courbes extrémales dans les espaces métriques fondés sur la notion d'aire. J. Math. Pures Appl. (9) 27, 255–336 (1948).

In a Riemannian space as well as in a Finsler space considered by Cartan [Les espaces de Finsler, Actual. Sci. Ind., no. 79, Hermann, Paris, 1934], the geodesic curves are auto-parallel and extremal curves of the curve length $\int ds$, but in the theory of a Cartan space, that is, of a space based on the notion of area [É. Cartan, Les espaces métriques fondés sur la notion d'aire, Actual. Sci. Ind., no. 72, Hermann, Paris, 1933] the auto-parallel curves are not extremal curves. Using Cartan's notations, the present author modifies the Euclidean connection Γ_{jk}^i of Cartan in a Cartan space by making the assumption that the torsion is $\Omega^k=-A_{i,r}J^r l^k l_h[dx^i dx^h]$ instead of Cartan's assumption $\Omega^k=0$, if the element of contact (or supporting element) (x, u) is transported parallel to itself along an infinitesimal cycle, where $A_{i,r}$ is the covariant derivative of the vector $A_i=gL^{-2}u_k u_h A^{hki}$:

$$A_{i,r}=\frac{\partial A_i}{\partial x^r}+\frac{\partial A_i}{\partial u^l}u_k\Gamma_{lr}^k-A_k\Gamma_{ir}^{*k}.$$

This Euclidean connection Γ_{jk}^i is to be determined from the system of equations

$$\text{(I)}\quad \begin{cases} \partial g_{ij}/\partial x^h=\Gamma_{jih}+\Gamma_{ijh}, \\ A_{i,r}J^r l^k l_h+\Gamma_{ih}^k+C_i^{ks}u_r\Gamma_{sh}^r=A_{h,r}J^r l^k l_i+\Gamma_{hi}^k+C_h^{ks}u_r\Gamma_{si}^r. \end{cases}$$

When the elements of contact along a curve are taken to be normal to the curve, the geodesic curves, i.e., auto-parallel curves in this connection, coincide with the extremal curves of the integral $\int ds$, whose equations are

$$K^{ij}\frac{\bar{\omega}^j}{ds}+A^{i,h}\frac{L^h}{\sqrt{g}}-\frac{g}{L^2}u_r u_s(\Gamma^{*rsi}-\Gamma^{*isr}) -\frac{\sqrt{g}}{L}u_r A_s(\Gamma^{*rsi}-\Gamma^{*isr})=0,$$

where

$$K^{ij}=A^{i:j}+2A_k A^{kij}+\frac{L^j}{\sqrt{g}}A^i+g^{ij},$$

$$A^{i:j}=\frac{L^2}{g\sqrt{g}}\frac{\partial^2\sqrt{g}}{\partial u_i\partial u_j}+\frac{L^j}{\sqrt{g}}A^i-2A^iA^j.$$

The following results for this connection are obtained. (1) The parameters Γ_{jk}^i are determined uniquely for a regular space, i.e., $|K^{ij}|\neq 0$. (2) A necessary and sufficient condition for the space to be totally singular, i.e., for the rank of the matrix (K^{ij}) to be 1, is $K^{ij}=L^iL^j/g$. (3) A necessary and sufficient condition that every element of contact determines one and only one extremal curve tangent to the normal unit vector of the element is $|K^{ij}|\neq 0$. (4) The multiple integral $\int F(x, u)dx^1 \cdots dx^{n-1}$, for which $|H^{ij}|$ or $|K^{ij}|$ is different from zero, admits a group of point transformations depending on at most $\frac{1}{2}n(n+1)$ parameters. (5) In a metric space two Euclidean connections with the same geodesic curves are identical. Applying the method of "repère mobile" of Cartan [La méthode du repère mobile, la théorie des groupes continus et les espaces généralisés, Actual. Sci. Ind., no. 194, Hermann, Paris, 1935], the author then obtains many geometrical results in the theory of curves and surfaces in this space, the results being generalizations of those for a Euclidean space, and discusses the equations of structure of infinitesimal displacement of the moving trihedral and the system of equations (I). Finally he calculates all quantities for some special spaces: the harmonic space $\iint(p^2+q^2)dxdy$ and the hyperharmonic space $\int(p_1^2+\cdots+p_n^2)dx^1 \cdots dx^n$, from which it can be concluded that the latter space is always regular while the former space is singular. [There are misprints on pp. 267–269.]

A. Kawaguchi (Sapporo).

Takasu, Tsurusaburo. Realisierung jeder von den elliptischen konformen, parabolischen konformen, hyperbolischen konformen, elliptischen Laguerreschen, parabolischen Laguerreschen und hyperbolischen Laguerreschen Räumen in einem andern. Tôhoku Math. J. 48, 331–343 (1941).

As defined by Scheffers, equilong geometry is the study of the group of all line transformations of the plane such that the distance between the points of contact of every two curves along a common tangent line is preserved. The equilong group consists of all monogenic functions of $z=x+py$ (or $\bar{z}=x-py$), where $p^2=0$, for which the first derivative does not vanish. The components of any dual complex number $z=x+py$ are the equilong coordinates (x, y) of a line. If (θ, P) are the Hessian coordinates of a line, that is, θ and P are the normal angle and normal distance of the line,

then $z=x+py=\tan(\theta+pP)/2=\tan\theta/2+p(P/2)\sec^2\theta/2$. The lines of the (θ, P)-plane are represented as points of the (x, y)-plane under the map $z=x+py=\tan(\theta+pP)/2$. Points and circles of the (θ, P)-plane are represented as vertical parabolas of the (x, y)-plane [see De Cicco, Bull. Amer. Math. Soc. **45**, 936–943 (1939); these Rev. **1**, 84].

The author extends the above results to any binary complex variable $z=x+my$ where m may be either i, or p, or h, for which $i^2=-1$, $p^2=0$, $h^2=+1$. The study of the m Laguerre transformations $w=(az+b)/(cz+d)$, where a, b, c, d are binary complex constants such that $ad-bc\neq 0$, is equivalent to the circular geometry in the elliptic, parabolic, and hyperbolic planes according as m is i, p, and h. That is, in the respective cases there are obtained the geometry of circles, the geometry of vertical parabolas, and the geometry of equilateral hyperbolas. By use of a transformation analogous to that used in the case of equilong geometry, the author considers these transformations as correspondences between the lines of a plane. The corresponding loci in the plane of points and in the plane of lines are studied. Finally, three-dimensional analogues are given.

J. De Cicco (Chicago, Ill.).

NUMERICAL AND GRAPHICAL METHODS

***Tables of Bessel Functions of Fractional Order. Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. Volume II.** Columbia University Press, New York, N. Y., 1949. xviii+365 pp. $10.00.

The purpose and scope of this work have been described in the review of vol. I [1948; these Rev. **9**, 533]. The present second volume parallels the first one and is devoted to the modified Bessel function of the first kind of orders $\pm 1/4$, $\pm 1/3$, $\pm 2/3$, $\pm 3/4$. It contains a note by R. E. Langer on the application of Bessel functions of fractional orders to the asymptotic representation of solutions of differential equations with large parameters. The introduction, by M. Abramowitz, gives a few necessary explanations, a note on interpolation by Everett's formulae, and an expansion of $x^{-\nu}I_\nu(x)$ (for arbitrary ν) in a series $\sum_{n=0}^{\infty} C_n I_{2n}(x)$ in which the coefficients C_n are obtained from a recurrence relation. A corresponding expansion for $x^{-\nu}J_\nu(x)$ is indicated (replace x by ix).

Tables: Constants (repeated from vol. I). $I_\nu(x)$ for $x=0(.001)X(.01)25.00$ to 10 D or 10 S (actually, $I_{-\nu}(x)$ is given up to $x=13$ only since $I_\nu(x)$ and $I_{-\nu}(x)$ coincide to 10 significant places for larger x); $X=1.00$ for $\nu=-3/4$, $-2/3$; $X=0.80$ for $\nu=-1/3$, $-1/4$; $X=0.60$ for $\nu=1/4$, $1/3$; $X=0.50$ for $\nu=2/3$, $3/4$. Tables of $e^{-x}I_\nu(x)$ for $x=25.0(.1)50(1)500(10)5000(100)10000(200)30000$ to 10 D; ν has the same values as above. In all these tables second order central differences (sometimes modified) and where necessary fourth order central differences are tabulated alongside the functions. Everett interpolation coefficients for second and fourth order central differences (the second order coefficients are now given to 10D as against 7D in the first volume, the fourth order coefficients to 6D as in vol. I). The table $L_\nu(\mu)$ is repeated from vol. I.

A. Erdélyi (Pasadena, Calif.).

***Tables of the Bessel Functions of the First Kind of Orders Forty Through Fifty-One, by the Staff of the Computation Laboratory.** The Annals of the Computation Laboratory of Harvard University, vol. XI. Harvard University Press, Cambridge, Mass., 1948. xi+620 pp. $10.00.

The present volume contains tables of $J_n(x)$ for $n=40(1)51$ and $x=0(.01)99.99$ to 10 decimals. *A. Erdélyi.*

Robbins, C. I., and Smith, R. C. T. A table of roots of $\sin Z=-Z$. Philos. Mag. (7) **39**, 1004–1005 (1948).

This gives six-decimal values of $Z=x+iy$ for the first 10 roots of $\sin Z=-Z$ with x and y positive. It is complementary to the similar table by Hillman and Salzer [same Mag. (7) **34**, 575 (1943); these Rev. **5**, 49] for the equation $\sin Z=Z$. *J. C. P. Miller* (London).

Hay, H. G. Five-figure table of the function

$$\int_0^\infty e^{-zy}\cdot \mathrm{Ai}^2(y-j_1)dy$$

in the complex plane. Philos. Mag. (7) **39**, 928–946 (1948).

In the function of the title, Ai (z), the Airy integral, is that solution of $d^2y/dx^2=zy$ which is real for real z and vanishes as $x\to\infty$, while $-j_1$ is the first zero of Ai (x). Values of the real and imaginary parts of $F(z)$ and $F'(z)$ are given to 5 decimals for $x=0(0.2)4$, $y=0(0.2)3.2$, $z=x+iy$. Values for $-y=0(0.2)3.2$ may be obtained by taking complex conjugates. Coupled differences $\frac{1}{2}(\Delta_x^2-\Delta_y^2)$ and $\Delta_x^2\Delta_y^2$, whose use is described by P. M. Woodward [same vol., 594–604 (1948); these Rev. **10**, 212], are provided for interpolation. *J. C. P. Miller* (London).

***Boll, Marcel. Remarques et Compléments aux Tables Numériques Universelles.** Dunod, Paris, 1949. 32 pp.

This pamphlet corrects very few of the exceedingly numerous errors in the author's Tables Numériques Universelles des Laboratoires et Bureaux d'Étude [Dunod, Paris, 1947; these Rev. **8**, 533] and adds four new tables. These tables are (1) $N^{\frac{1}{2}}$, $N=[1(1)31; 6D]$, $N=[32(1)1000; 5D]$; (2) the involute function, $\tan x-x$, for $x=[1°(1°)60°; 6D]$; and two corrected tables replacing former similar tables; (3) prime numbers and smallest divisors, pp. 62–70; (4) Fresnel integral tables, from uniformly 4D, pp. 362–363, to tables with many 5–6D values. The pamphlet is distributed free to all purchasers of the original volume who apply for it. *R. C. Archibald* (Providence, R. I.).

West, C. F., and DeTurk, J. E. A digital computer for scientific applications. Proc. I.R.E. **36**, 1452–1460 (1948).

During the past two years development has been initiated on several large-scale automatic digital computing machines. The present paper is concerned with the over-all organization of one such machine. A logical division of the machine into four major components is described, and the machine performance is interpreted in terms of these component functions. The electronic techniques used to accomplish the storage, transmission, and arithmetic manipulation of numbers, together with certain methods used for control of the computer, are briefly discussed.

From the authors' summary.

***Milne, William Edmund. Numerical Calculus. Approximations, Interpolation, Finite Differences, Numerical Integration, and Curve Fitting.** Princeton University Press, Princeton, New Jersey, 1949. x+393 pp. $3.75.

From the author's preface. This book "is designed to provide rudimentary instruction in such topics as solution

of equations, interpolation, numerical integration, numerical solution of differential equations, finite differences, approximations by least squares, smoothing of data, and simple equations in finite differences. The presentation is intentionally elementary so that anyone with some knowledge of calculus and differential equations can read it understandingly. Mathematical elegance and rigor have frequently been sacrificed in favor of a purely naïve treatment." *L. M. Milne-Thomson* (Greenwich).

Simonsen, W. On divided differences and osculatory interpolation. Skand. Aktuarietidskr. **31**, 157–164 (1948).

This paper gives a new derivation of Hermite's interpolation formula [J. Reine Angew. Math. **84**, 70–79 (1877)], which, however, the author ascribes to Johansen [Skand. Aktuarietidskr. **14**, 231–237 (1931)]. Incidentally, he obtains a formula previously given by Milne-Thomson [The Calculus of Finite Differences, Macmillan, London, 1933, p. 14] for a general divided difference with repeated arguments in terms of a divided difference with distinct arguments.

T. N. E. Greville (Washington, D. C.).

White, Aubrey. Interlocking interpolation curves. Trans. Actuar. Soc. America **49**, 337–364 (1948).

The author develops a family of smooth-junction interpolation formulas using spliced polynomial arcs, but without imposing conditions on the values of the derivatives at the points of junction of adjacent arcs, requiring instead that adjacent arcs have identical ordinates at certain specified points in the neighborhood of the points of junction. The formulas obtained may be regarded as the finite difference analogues of the traditional osculatory and "semi-osculatory" formulas. *T. N. E. Greville* (Washington, D. C.).

Salzer, Herbert E. Coefficients for facilitating trigonometric interpolation. J. Math. Physics **27**, 274–278 (1949).

Gauss's formula for nth order trigonometric interpolation of a function $f(x)$ which assumes the values $f_i=f(x_i)$, $i=0, 1, 2, \cdots, 2n$, x_i equidistant, can be given the form $f(x)=\sum_i a_i^{(2n+1)} f_i/\sum_i a_i^{(2n+1)}$, where

$$a_i^{(2n+1)}=A_i^{(2n+1)}/\sin \tfrac{1}{2}(x-x_i).$$

To facilitate the computation the author has tabulated to 8 significant figures the numerical coefficients

$$A_i^{(2n+1)}=1/\prod_j \sin \tfrac{1}{2}(x_i-x_j)$$

for several values of i and n. *E. Bodewig* (The Hague).

Kuntzmann, Jean. Formules de quadrature approchée pour les fonctions continues à dérivée première continue et à dérivée seconde bornée. C. R. Acad. Sci. Paris **228**, 38–40 (1949).

The author continues his former studies [same C. R. **227**, 584–586 (1948); these Rev. **10**, 330]. The continuous curves $f(x)$ with $f'(x)$ continuous and $|f''(x)|<K$ lie between two curves formed by parabolic arcs. The best integration formulas are always those where the n given points have the same distance h and lie symmetrically in the integration interval $(-L, +L)$. Several such formulas arise. Thus the integral is approximately $L[f(L)+f(-L)]+\frac{1}{3}L^2[f'(L)-f'(-L)]$ with a maximum error of $\frac{1}{3}KL^3$. Another formula for the integral is area of trapezoid$+(3h^2/32)[f'(L)-f'(-L)]$.

E. Bodewig (The Hague).

Kopal, Zdeněk. A table of the coefficients of the Hermite quadrature formula. J. Math. Physics **27**, 259–261 (1949).

For the approximate formula $\int_{-\infty}^{\infty} e^{-x^2}f(x)dx=\pi^{\frac{1}{2}}\sum_{i=1}^{n} p_i f(x_i)$, where $p_i=2^{n+1}n!/[H_n'(x_i)]^2$, $H_n=e^{x^2}D^n(e^{-x^2})$, H_n the Hermite polynomials, x_i the roots of H_n, the author gives the values of the p_i for $n=10(1)20$ to at least 6 decimal places together with the x_i to 6 decimal places. *E. Bodewig*.

Lyusternik, L. A., and Ditkin, V. A. Approximate formulas for the calculation of multiple integrals. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1163–1168 (1948). (Russian)

Another version of an article already reviewed [Doklady Akad. Nauk SSSR (N.S.) **61**, 441–444 (1948); these Rev. **10**, 153]. *E. M. Bruins* (Amsterdam).

Rose, A.-J. Machine à calculer permettant la détermination de fonctions périodiques et leur introduction dans des calculs. Application à la sommation des séries de Fourier et au calcul des facteurs de structure en cristallographie. J. Recherches Centre Nat. Recherche Sci. **2**, 139–144 (1948).

The electronic density ρ at the point x, y, z of a crystal structure is given by

$$\rho=\sum_h \sum_k \sum_l [\pm A(hkl) \cos 2\pi(hx+ky+lz)],$$

where h, k, l are the integral indices of the plane grid and A is the amplitude of the bundle of rays diffracted by the plane. This computation is mechanized by the use of the Thomas cylinder (a long cylindrical gear wheel) on which the number of teeth removed at each integral distance along the axis is proportional to this distance. A pinion which engages the Thomas cylinder at the distance x is rotated an amount proportional to hx as the cylinder is rotated h times. For a fixed l, the sum for varying h and k, obtained from two Thomas cylinders, drives a drum graduated in the cosine function. Multiplication by $A(hkl)$ is obtained by the use of another Thomas cylinder. The products so obtained are registered for the several values of l and then summed. A prototype of this mechanism has been constructed at the Laboratoires de Bellevue. An electrical analogue, using brushes which make contact with conducting spots on an insulating cylinder, is described.

M. Goldberg (Washington, D. C).

Simon-Suisse, J., Bourcier de Carbon, Ch., et Mazet, R. Sur la recherche des racines réelles ou complexes des équations algébriques. Recherche Aéronautique **1948**, no. 5, 53–61 (1948).

The author describes two methods for finding the roots of algebraic equations. First, that of finding the characteristic roots of the companion-matrix; second, a semi-graphical procedure which is identical to that mentioned by the reviewer [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 218–223 (1947); these Rev. **10**, 405].

E. Bodewig (The Hague).

Romanini, Clelia. Sulla risoluzione grafica delle equazioni di 5° e 6° grado. Period. Mat. (4) **26**, 81–101 (1948).

After describing the two methods of Descartes for solving fifth and sixth degree equations graphically and the method of Amseder for fifth degree equations, the author gives two new methods for solving the reduced fifth and sixth degree equations. The first consists in projecting the intersections

of the fixed plane cubic curve $y^2=x^3$ with the variable hyperboloid $x(cx+dy)+ax+by+1=0$ from the origin on the line $y=1$. The solution of $x^6+ax^4+bx^3+cx^2+dx+e=0$ is furnished by the intersections of the fixed cubic skew curve $y=x^2$, $z=x^3$ with the variable quadric

$$z^2+a(x^2+y^2)+dx+(c-a)y+bz+e=0.$$

The points are found by means of descriptive geometry.

E. Bodewig (The Hague).

Kivikoski, E. Über die Konvergenz des Iterationsverfahrens bei der Berechnung des effektiven Zinsfusses der Anleihen. Skand. Aktuarietidskr. **31**, 135–156 (1948).

The determination of the effective rate of interest for a loan requires the solution of an algebraic equation. In general, the degree of this equation is rather high so that it is convenient to obtain an approximate solution. This is often done by iteration. H. Holme [Skand. Aktuarietidskr. **15**, 225–250 (1932)] proposed an iterative method. However, R. von Mises [Skand. Aktuarietidskr. **16**, 229–231 (1933)] showed that this iteration does not always converge if $k<1$. Here k denotes the issue price of the loan. If $k>1$ the iteration discussed by Holme converges. The author modifies the formula used by Holme and obtains an iteration which converges if $k<1$ but does not always converge if $k>1$. He also discusses cases where the process is convergent although $k>1$. *E. Lukacs* (China Lake, Calif.).

Stein, P., and Rosenberg, R. L. On the solution of linear simultaneous equations by iteration. J. London Math. Soc. **23**, 111–118 (1948).

The authors study the convergence of the Seidelian iteration O and the usual iteration S. In both cases the errors of the nth approximation vector are proportional to the nth power of a matrix. In discussing the convergence of the powers of a matrix the authors start from some fundamental theorems of Frobenius and give some other known theorems [the first of which is not due to Berry, but to E. Schmidt] and some new ones. Theorem III states that A^n converges to zero if $a-bq<(1-c)(q-1)$, where $q=(a/b)^{1/m}$, m is the order of A, and a, b, c, respectively, are the moduli of the dominant elements of A below, above, or in the diagonal. Theorem VI states that iteration O converges better than S if in the system $(E-C)x=d$ of equations the elements of C are nonnegative. This theorem does not hold if C contains positive and negative elements, so that in this case the convergence of S is independent of that of O. Furthermore, the order in which the equations are taken is important for the convergence of O. Theorem VII says that if the elements of C are nonnegative the convergence of O will be better if the equations are so ordered that the smaller elements of C lie above the diagonal. *E. Bodewig.*

Stefaniak, H. St. Eine einfache Methode zur Ermittlung der charakteristischen Daten eines gedämpft schwingenden Systems zweiter Ordnung mit Hilfe einer neuen Auftragung der Resonanzkurven. Z. Angew. Math. Mech. **28**, 368–371 (1948).

Charts are described which facilitate the solution of problems on one-dimensional forced oscillations.

P. Franklin (Cambridge, Mass.).

Djang, Gwoh-Fan. A modified method of iteration of the Picard type in the solution of differential equations. J. Franklin Inst. **246**, 453–457 (1948).

The author is concerned with Picard's method of iteration for the solution of the nth order equation

$$y^{(n)}=f(y^{n-1},\cdots,y^{(1)},y;x). \tag{1}$$

He illustrates with three examples of second order equations that it is often worth while to choose the initial trial solution as the exact solution of an analytically tractable approximation to equation (1). *H. O. Hartley* (Princeton, N. J.).

Matthieu, P. Über das Iterationsverfahren von Picard-Lindelöf zur angenäherten Lösung gewöhnlicher Differentialgleichungen. Elemente der Math. **4**, 34–42 (1949).

Expository article.

Mikeladze, Š. E. New formulas for the numerical integration of differential equations. Doklady Akad. Nauk SSSR (N.S.) **61**, 789–790 (1948). (Russian)

The interpolation formula of Newton for $y(t)$ can be used not only with increasing abscissas $t_1<\cdots<t_n$, but also with zigzagging abscissas $t_1,\cdots,t_n$. Representing $y(a+ht_\beta)$ by the Taylor formula with the initial value $y(a+ht_\alpha)$ combined with a development of the integral for the remaining term according to Newton's interpolation formula with the abscissas $a+ht_i$, several different formulae can be obtained by permutation of the t_i. For example, $t_\alpha=0$, $t_\beta=1$; $a, a+h, \cdots, a+rh$ gives $y^{(k)}(a+h)$ expressed by $y^{(p)}(a)$ and the mean kth order differences of $y^{(p)}(a+ht_i)$, $p\geqq k$; $t_\alpha=1$, $t_\beta=2$; $a+h, a, a+2h, \cdots, a+rh$ gives $y^{(k)}(a+2h)$; $t_\alpha=2$, $t_\beta=3$; $a+2h, a+h, a, a+3h, \cdots, a+rh$ gives $y^{(k)}(a+2h)$, etc., in an analogous form. *E. M. Bruins* (Amsterdam).

Carter, A. E., and Sadler, D. H. The application of the National accounting machine to the solution of first-order differential equations. Quart. J. Mech. Appl. Math. **1**, 433–441 (1948).

The authors describe in detail a mechanical process on the "National accounting machine" for solving the first order differential equation $y'=f(x,y)$. The method is a numerical step by step integration at equidistant argument h and is based on "Milne's formula"

$$y_n-y_{n-4}=\left(3g+2\delta^2+\frac{7}{30}\delta^4-\frac{2}{315}\delta^6+\cdots\right)_{n-2}, \tag{1}$$

where $y_n=y(nh)$, $g=\frac{4}{3}f$ and the δ's are central differences of g. At the stage of the operation when all values up to y_{n-1} are known an approximate value of y_n is computed on the National from (1) using the known g and δ^2, an extrapolation estimate of $\frac{7}{30}\delta^4$ and ignoring δ^6. The approximate y_n so obtained is used for computation of $g(x_n, y_n)$ (on an auxiliary machine) from which a corrected δ^4 is obtained resulting in turn in corrections to y_n, $g(x_n, y_n)$ and δ^4 itself. The value of y_n is corrected by hand and the correct δ^4 fed into the National machine where the cycle is completed producing the correct value of $g(x_n, y_n)$ and an approximate value of y_{n+1} from (1). The process can be adjusted to allow for δ^6. The formula (1) is a compromise between the slowly convergent "backward difference" formulae and the central difference process requiring excessive estimation. The iterative correction is here carried out at one stage of the cycle of operation and the contents of the National registers need not be altered. The method does not appear worth while for solutions to low digital accuracy. *H. O. Hartley.*

Fox, L. The solution by relaxation methods of ordinary differential equations. Proc. Cambridge Philos. Soc. **45**, 50–68 (1949).

The aim of this paper is to demonstrate the application of relaxation methods to the solution of ordinary differential equations in which boundary conditions are specified at two points, the ends of the range of integration. It is shown by a series of examples, e.g., the calculation of the Bessel function $J_0(x)$ to 8 decimal places at intervals of 0.1 in the range $4 \leqq x \leqq 5$, given $J_0(4)$ and $J_0(5)$; the calculation of the deflection y of a beam of flexural rigidity B, clamped at its ends $x=0$ and $x=1$, and bent by a distributed load w; and an eigenvalue problem, that the precision obtainable can be judged as in any well-established computational technique, and that statements such as "the error in the last figure is less than one unit" can be made with this method as with others. *S. C. van Veen* (Delft).

Ferretti, B., and Krook, M. On the solution of scattering and related problems. Proc. Phys. Soc. **60**, 481–490 (1948).

For calculating scattering cross-sections it is not necessary to know the detailed solution of the relevant wave-equation, but only its asymptotic form. Thus for a system with forces of finite range a, the scattering problem is effectively solved when the logarithmic derivatives of the interior radial solutions at the boundary a are determined. The object of this paper is to present a method of evaluating the logarithmic derivative of the interior solution at a without solving the equation explicitly. The method is based on a Taylor expansion about a. In general the radial differential equation is linear, of the second order and with the origin as a regular singular point. The Taylor series about the ordinary point a is $u(r)=\sum_{n=0}^{\infty} c_n(r-a)^n$; c_n can be represented in the form $c_n=F_n c_0+G_n c_1$ $(F_0=G_1=1,\ F_1=G_0=0)$. The coefficients F_n, G_n $(n=2, 3, \cdots)$ are uniquely determined by the differential equation. The main theorem is as follows. The logarithmic derivative $u'(a)/u(a)=b$ of the solution which is regular at $r=0$ is given by (1) $b=-\lim_{n\to\infty} F_n/G_n$ if the origin is the singular point of the differential equation nearest to a. When the distance from $r=a$ to the next nearest singularity is d, and a/d is not too close to unity, the convergence in (1) to the limit b is shown to be rapid. The procedure is illustrated by a number of examples. The author remarks that by using the approximate value $b_n=-F_n/G_n$ an approximate representation of $u(a)$ over a fairly wide range may be obtained, in many cases in terms of elementary functions. The constant of integration can be fixed by using the known behaviour of $u(a)$ in the neighbourhood of the origin.

The method is also applicable to problems in which the required solution is not regular at the origin. In this case the method selects the "least singular" solution consistent with the specified data. A proof of this, in a form sufficiently general for the purpose of this article, is given in the appendix. *S. C. van Veen* (Delft).

Duncan, W. J. Assessment of errors in approximate solutions of differential equations. Quart. J. Mech. Appl. Math. **1**, 470–476 (1948).

The author gives a concise account of three methods of gauging the accuracy of approximate numerical solutions of differential equations. (i) If Δ is a linear differential operator and if the Green's function of the boundary problem is positive, if further ϕ and ϕ_a are respectively the solutions of $\Delta\phi=f$ and $\Delta\phi=f+\epsilon$, then $\phi_a-\sigma|\epsilon|_{\max} \leqq \phi \leqq \phi_a-\sigma|\epsilon|_{\min}$, where $\Delta\sigma=1$ and ϕ, ϕ_a and σ satisfy the boundary conditions. Negative Green's functions are similarly treated. (ii) Restatement of L. F. Richardson's "deferred approach to the limit" for an approximate process of order $k-1$. (iii) Generalisation of (i) when an approximate Green's function is used. *H. O. Hartley* (Princeton, N. J.).

Snyder, Frances E., and Livingston, Hubert M. Coding of a Laplace boundary value problem for the UNIVAC. Math. Tables and Other Aids to Computation **3**, 341–350 (1949).

The UNIVAC is a proposed electronic computer. In this paper a brief description of the computer and its instruction code has been given. "The solution of the plane potential problem using the Liebmann method of finite differences has been formulated in terms of the instruction code for the UNIVAC computer. Explicit coding has been given only for the simple rectangular boundary, but the nature of the modifications required for more general boundaries has been discussed." *R. W. Hamming* (Murray Hill, N. J.).

Epstein, Bernard. A method for the solution of the Dirichlet problem for certain types of domains. Quart. Appl. Math. **6**, 301–317 (1948).

L'auteur donne une méthode de résolution du problème de Dirichlet (pour les fonctions harmoniques de deux variables) dans un domaine S qui soit la réunion de deux domaines S_1, S_2 ayant en commun une partie de frontière, lorsqu'on sait résoudre le problème dans S_1 et dans S_2. On peut considérer la méthode comme un cas limite du procédé alterné de Schwarz (qui s'applique lorsque S_1, S_2 ont en commun des points intérieurs) sous la forme que lui a donnée R. Nevanlinna [J. Reine Angew. Math. **180**, 121–128 (1939)]. En prenant pour inconnue la fonction $f(s)$ qui donne les valeurs de la fonction harmonique sur la partie de frontière commune à S_1, S_2 (que nous indiquerons ici par Γ) et en appliquant le théorème de la moyenne de Gauss dans un cercle ayant le centre dans un point $P(s)$ de Γ et un convenable rayon $R(s)$, l'auteur écrit pour $f(s)$ une équation intégrale de deuxième espèce, dont le noyau $K_R(s, s')$ peut s'exprimer par les fonctions de Green G_1, G_2 des domaines S_1, S_2. Il résout cette équation intégrale, par approximations successives, dans l'hypothèse qu'on peut choisir la fonction $R(s)$ de telle sorte que l'intégrale $\int_\Gamma K_R(s, s')ds'$ (qui est toujours positive et non supérieure à 1) soit non supérieure à $p<1$. Il indique des classes de domaines pour lesquels une telle choix est possible. La méthode est appliquée à des cas particuliers. Il y a aussi des exemples numériques. *A. Ghizzetti* (Pisa).

Blenk, H. Nomogramme für die Gleichung 4. Grades mit reellen oder komplexen Wurzeln. Z. Angew. Math. Mech. **29**, 58–61 (1949).

Morgantini, Edmondo. Sulle equazioni in sei variabili rappresentabili con un nomogramma a punti allineati. Rend. Sem. Mat. Univ. Padova **17**, 115–138 (1948).

Let F be a given function of the six variables $t_1, \cdots, t_6$ and D a determinant of order three, whose rows are composed respectively of functions of t_1, t_2, of t_3, t_4 and of t_5, t_6. The paper gives sufficient conditions that the equation $F=0$ can be written in the form $D=0$ and hence represented by a nomogram. *J. M. Thomas.*

Zwinggi, Ernst. Initiation of a formula for approximate valuation of premiums for disability benefits. Skand. Aktuarietidskr. **31**, 165–170 (1948).

The author considers the premium for a disability pension contracted by a person in the active state. The annuity is due if invalidity occurs within a certain period and terminates at the end of the same period. The premiums are payable during this period as long as the person insured is active. The premium for this insurance is often approximated by substituting the life table for the table of active persons. The author examines this approximation and gives an interpretation for the difference between the approximation and the exact value. *E. Lukacs* (China Lake, Calif.).

Wolf, Helmut. Über eine allgemeine Form der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Naturf. Ges. Bamberg. Ber. **31**, 41–45 (1948).

Jensen, Henry. On the superposition of the differential-equations of the geodetic line. With a calculation-example. Mém. Inst. Géodésique Danemark [Geodaetisk Instituts Skr.] (3) **13**, 23 pp. (1948).

ASTRONOMY

Hil'mi, G. F. On the possibility of capture in the problem of three bodies. Doklady Akad. Nauk SSSR (N.S.) **62**, 39–42 (1948). (Russian)

Šmidt, O. Yu. The theory of capture and statistical laws of the distribution of the orbits of double stars. Doklady Akad. Nauk SSSR (N.S.) **62**, 43–46 (1948). (Russian)

Šmidt, O. Yu., and Hil'mi, G. F. The problem of capture in the three body problem. Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 157–159 (1948). (Russian)

A solution of the three-body problem can be said to represent a capture if two of the bodies separate to infinite distance in the past but remain within a constant distance in the future. This definition was replaced by Šmidt by the following one. The solution represents capture if at a time t_1 the three pairs of relative motions (regarded as initial conditions for the two-body problem) are hyperbolic and the distances all exceed a constant R_1, while at a later time t_2 two of the bodies are within distance $R_2<R_1$ and have elliptic motions, while the third body is at a distance greater than R_1 and has a hyperbolic motion. The constants R_1 and R_2 are interpreted respectively as a mean stellar distance in the galaxy and a mean diameter of a double star. In a previous paper [Doklady Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947)] Šmidt gave a numerical example to show the possibility of such capture. Hil'mi generalizes this by requiring (I) existence of a time t_1 such that $r'<-Br^{\frac{1}{2}}$, where r is the minimum distance between the bodies, r' is the minimum of the three radial velocities, and B is a certain function of the masses, and (II) of a time t_2 such that for some R and $\epsilon<R$ one has $2r<2R<\rho$, $\rho'>0$ and $\rho'-A\rho^{-1}>C$. Here A and C are certain functions of the masses, R, ϵ and h (the energy constant), while ρ is the distance between the furthest body and the center of mass of the two close ones, and $\rho'=d\rho/dt$. Hil'mi states the following theorems. (1) If condition (I) holds, then all three distances become infinite as t approaches $-\infty$. (2) If $h>0$ and (II) holds, then ρ increases steadily to ∞ as t increases, while $r\leqq R$ for $t>t_2$. (3) Šmidt's example satisfies (I) and (II), hence illustrates capture in terms of Hil'mi's definition. (4) The measure of the set of initial conditions leading to capture is not zero. The proof of the last theorem is given in a few lines.

The results stated are given in the first of the three papers, and their astronomical significance for double stars and for the solar system is discussed briefly in the second and third papers. The third paper also contains a discussion of the probability distribution of the semi-axis a for double stars. On the basis of rough approximations and simple analysis it is concluded that the probability density should be of the form $(a+\rho)^{-2}$ for large a, where ρ is a constant. *W. Kaplan* (Zurich).

Agostinelli, Cataldo. Interpretazione elettrodinamica della legge di attrazione universale e nuova spiegazione dello spostamento del perielio di Mercurio. Atti Sem. Mat. Fis. Univ. Modena **1**, 148–164 (1947).

Colacevich, A. Relazioni analitiche tra le distribuzione delle velocità lineari delle stelle. Osservazioni e Memorie dell'Osservatorio Astrofisico di Arcetri, no. **64**, 3–24 (1 plate) (1947).

It is assumed that the frequency distribution function of the peculiar space motions of the stars is spherically symmetrical and vanishes for large values of the velocities. The function is considered to be the same throughout space and the stars are taken to be evenly distributed over the sky. Analytical relationships are then derived expressing the distribution in radial and tangential velocity as separate functions of the distribution in space velocity for the case of both an observer at rest and in uniform motion with respect to the field of peculiar velocities. In the latter case formulae are also given for the distribution of tangential velocity along a meridian and parallel. The case of a Gaussian distribution in radial velocity is worked out. *B. J. Bok* (Cambridge, Mass.).

White, Marvin Lee. Dynamical friction. Astrophys. J. **109**, 159–163 (1949).

In Chandrasekhar's original papers on dynamical friction, an approximation was made which did not fully account for the effect of encounters with field stars the velocities of which are in excess of those of the star under consideration. Exact calculations show that, for large values of the velocity of the field star, the coefficient of dynamical friction is somewhat larger than was given by the earlier calculations. The half-life of a cluster like the Pleiades is thus decreased by 15%, but this change is hardly an appreciable one and the estimated half-life of the Pleiades still stands at approximately 3×10^9 years. *B. J. Bok* (Cambridge, Mass.).

ten Bruggencate, P. Zur Gestalt von Spiralnebeln. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1948**, 1–7 (1948).

The author discusses the properties of nonstationary stellar systems characterized at any point by an ellipsoidal distribution of velocities and exhibiting differential star streaming. The field of force is assumed to be due to a small central nucleus surrounded by a highly flattened spheroid

of smaller mean density, and to contain an arbitrary time function. Transformations are outlined which permit one to effect a transition from nonstationary to stationary fields of force. In particular, it is shown that circular orbits in a stationary field of force degenerate into logarithmic spirals in nonstationary fields. The spiral arms of a galaxy are regarded as loci of points which the stars pass through preferentially in the plane of symmetry of nonstationary systems. The author has applied his theory to explain the observed form of the spiral arms of the neighboring nebula Messier 33, and satisfactory agreement is claimed. A hypothesis is proposed that the stars of Baade's "population I" are describing logarithmic spirals in a nonstationary field of force invoked by the stars of the "population II."

Z. Kopal (Cambridge, Mass.).

von Weizsäcker, Carl Friedrich. Die Rotation kosmischer Gasmassen. Z. Naturforschung **3a**, 524–539 (1948).

On the basis of Prandtl's theory, the hydrodynamical equations of motion for a turbulent gas are derived de novo. The effect of turbulent pressure upon stability criteria is studied. The equations are first solved for the case of a two-dimensional system with rotational symmetry. Because of turbulent friction a rotating mass is found to separate in a nucleus with a surrounding slowly dissipating atmosphere. Under certain conditions "Saturn-like" ring systems may develop. The nucleus will, to a first approximation, show a solid-body rotation. The three-dimensional case is considerably more complex, but there is no suggestion of any basically different conclusions.

These considerations provide an explanation for Babcock's observed law of rotation in the Andromeda Nebula. The author accepts as essentially correct the derived mass distribution of Wyse and Mayall (a dense nucleus, then a zone of relatively low density and finally a broad outer ring of higher density). The outer parts of the nebula should expand at a rate of 2 to 3 km/sec, slow enough to permit the nebula a life-time of the order of at least several times 10^9 years.

According to the author and W. Heisenberg, the spiral structure may be caused by distortions of the newly-formed clouds as a consequence of differential rotation. If this interpretation is correct, Babcock's results on rotation would suggest spiral arms running in opposite senses in the inner and outer parts of the Andromeda Nebula. Published photographs lack the needed high definition to decide this point.

B. J. Bok (Cambridge, Mass.).

MECHANICS

***Milne, E. A. Vectorial Mechanics.** Interscience Publishers Inc., New York, N. Y., 1948. xiii+382 pp. $7.50.

The reviewer has long felt that the role of vector analysis in mechanics has been much overemphasized. It is true that the fundamental equations of motion in their various forms, especially in the case of rigid bodies, can be derived with greatest economy of thought by use of vectors (assuming that the requisite technique has already been developed); but once the equations have been set up, the usual procedure is to drop vector methods in their solution. If this position can be successfully refuted, this has been done in the present work, the most novel feature of which is to solve the vector differential equations by vector methods without ever writing down the corresponding scalar differential equations obtained by taking components. The author has certainly been successful in showing that this can be done in fairly simple, though nontrivial, cases. To give an example of a definitely nontrivial problem solved in this way, one might mention the nonholonomic problem afforded by the motion of a sphere rolling on a rough inclined plane or on a rough spherical surface. The author's methods are interesting and aesthetically satisfying and therefore deserve the widest publication even if they partake of the nature of a tour de force.

The book contains a complete treatment of vector analysis, including the divergence and curl theorems, which of course are not really needed in the dynamical portion of the book in which attention is confined to particles and rigid bodies. A certain amount of tensor analysis is also included, but, since the only ground metric employed is the usual Euclidean one in Cartesian coordinates, the distinction between covariance and contravariance is omitted. The first 160 pages, including a ten page chapter on statics, are thus devoted to this preliminary acquisition of technique in vector analysis before dynamics, or even kinematics, is introduced. The intrinsic beauty of this technique, if not its actual usefulness, is ample justification for this amount of space.

The book contains a very careful exposition of the fundamental notions of force and mass. It would have been desirable, however, if the undefined concept of an unaccelerated frame of reference had been inserted before the definition of zero force instead of afterwards. Similarly satisfying is the treatment of the angular velocity of a rigid body which is certainly far superior to what is usually found in most other treatments. However, one cannot help remarking that the formula for angular velocity [p. 163], $\Omega=(\dot{r}_1\times\dot{r}_2)/(\dot{r}_1\cdot r_2)=(\dot{r}_2\times\dot{r}_1)/(\dot{r}_2\cdot r_1)$, in terms of the position vectors r_1 and r_2 of two points in the rigid body, becomes indeterminate when $\dot{r}_1$ is perpendicular to r_2. This case seems not to have been excluded by the author, although he was careful to indicate that r_1 and r_2 were supposed to be non-parallel.

The author states in his preface that "no claim is made for originality of results" and that "the methods of analytical dynamics (Lagrange, Hamilton) have been deliberately excluded, so as to maintain the elementary character of the work."

D. C. Lewis (Baltimore, Md.).

***Timoshenko, S., and Young, D. H. Advanced Dynamics.** McGraw-Hill Book Company, Inc., New York, 1948. xii+400 pp. $5.50.

From the mathematical point of view the title of this book is a misnomer. The mathematics is entirely elementary, the most advanced topics being Lagrange's equations and Hamilton's principle. Vector notation is not used. Even the algebraic problem of the simultaneous reduction of two symmetric matrices to diagonal form (or the corresponding problem formulated in terms of quadratic forms) is omitted in the chapter of some 80 pages devoted to the theory of small oscillations. The merit of the book consists in the vast wealth of problems, many of which are worked out in the text in illustration of fundamental principles. The emphasis is on problems in engineering including methods of numerical computation, but the astronomical problem of two bodies is not omitted. Also included is a discussion of the possi-

bility of sending a rocket to the moon. There is a great deal on the gyroscope and its applications, but nothing about the solution of such problems in terms of elliptic functions. *D. C. Lewis* (Baltimore, Md.).

Moritz, E. Vom Parallelogramm der Kräfte. Optik **3**, 96–100 (1948).

The law of combining forces is deduced from an assumption of symmetry for the case of equal forces, combined with the assumed regularity of a numerical function of angle, eventually shown to be the cosine. *P. Franklin.*

Kosticyn, V. T. On the minimum dimensions of cam mechanisms. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1531–1537 (1948). (Russian)

A cam of minimum radius R and maximum radius $R+H$ actuates as a follower a rod sliding between two guide rings $2b$ apart, through the intermediary of a roller caster of radius r. The problem is to find the minimum value of the dimensions $2(H+R+b)$ of the casing, for a constant swing H. This lower bound arises from the condition that self-locking is to be prevented, or that the angle γ between the bar axis and the cam normal is to be smaller than a prescribed quantity smaller than the critical value $\tan^{-1} b/f(H+r+b)$ at which the reaction at the center of the caster wheel becomes infinite (f is the coefficient of friction) The product of f and the diameter of the bar is neglected. A relation between R and b is obtained from the value of the maximum of γ, and is used to find the minimum value of $R+b$ in terms of H, r, f for a certain special prescribed relation between cam and slider motion.

A. W. Wundheiler (Chicago, Ill.).

Pailloux, Henri. Sur quelques problèmes d'oscillations. C. R. Acad. Sci. Paris **227**, 1208–1210 (1948).

Considering the oscillation of a hanging cord, the author indicates the conditions under which the solution to a certain boundary value problem exists. A power series technique is used and the convergence questions are omitted. A second problem deals with a dissipative oscillat'ng system. For a very special case, the problem is reduced to that of solving some algebraic equations. *G. F. Carrier.*

Pailloux, Henri. Sur les petits mouvements verticaux d'un fil pesant. C. R. Acad. Sci. Paris **228**, 808–810 (1949).

The linear equations for the small disturbances in a vertical plane of a string from the equilibrium catenary are set up and solved for certain limiting cases.

P. Franklin (Cambridge, Mass.).

Schäfke, Friedrich Wilhelm. Über die Wirkung der drei reinen Kopplungsarten zweier frei schwingenden Systeme. Math. Nachr. **1**, 66–80 (1948).

This paper relates to the theory of linear dynamical systems with two degrees of freedom, having systems of differential equations of motion of the form

$$x_1''+2e_1x_1'+k_1^2x_1+\rho_1x_2''+2\sigma_1e_1x_2'+\tau_1k_1^2x_2=0,$$
$$x_2''+2e_2x_2'+k_2^2x_2+\rho_2x_1''+2\sigma_2e_2x_1'+\tau_2k_2^2x_1=0,$$

the coefficients being real constants such that

$$\frac{\rho_2}{\rho_1}=\frac{2\sigma_2e_2}{2\sigma_1e_1}=\frac{\tau_2k_2^2}{\tau_1k_1^2}>0.$$

It is assumed that the parameters $\rho^2=\rho_1\rho_2$, $\sigma^2=\sigma_1\sigma_2$, $\tau^2=\tau_1\tau_2$ satisfy the relations $0\leqq\rho^2<1$, $0\leqq\sigma^2<1$, $0\leqq\tau^2<1$. The author studies the behavior of the roots of the characteristic equation

$$\begin{vmatrix}\lambda^2+2e_1\lambda+k_1^2 & \rho_1\lambda^2+2\sigma_1e_1\lambda+\tau_1k_1^2\\ \lambda^2+2e_2\lambda+k_2^2 & \rho_2\lambda^2+2\sigma_2e_2\lambda+\tau_2k_2^2\end{vmatrix}=0$$

when one of the parameters ρ, σ, τ is varied while the other two are held fixed at the value zero. He also discusses some of the physical implications of the results of this study. The results given in the paper consist essentially of a large body of diverse details, and they do not admit of any brief summary. *L. A. MacColl* (New York, N. Y.).

Chaléat, Raymond. Sur un dispositif de couplage de plusieurs pendules. C. R. Acad. Sci. Paris **228**, 538–540 (1949).

A case, stable in theory, for two pendulums coupled by an electromagnetic mechanism is discussed.

P. Franklin (Cambridge, Mass.).

Bordoni, P. G. Moti alla Cardano di un bipendolo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 147–154 (1948).

Schürer, Fritz. Zur Theorie des Balancierens. Math. Nachr. **1**, 295–331 (1948).

The equation governing regulated forced oscillations is $f''(t)=af(t)-hf'(t)-bf(t-\delta)-cf'(t-\delta)+g(t)$. The solution is a series of exponentials, obtained from a characteristic transcendental equation. Stable solutions result when all the characteristic roots have negative real part, and the author obtains simple conditions for this.

P. Franklin (Cambridge, Mass.).

Valentine, F. A. The motion of a sliding horizontal hoop. Amer. Math. Monthly **56**, 79–87 (1949).

By analytical reasoning on inequalities and comparison equations, qualitative results are deduced for a problem in mechanics whose equations do not admit of an elementary solution. *P. Franklin* (Cambridge, Mass.).

Stange, K. Über die Bewegung eines stabilen schweren symmetrischen Kreisels bei kleinen Störungen des Längsschwunges. Ing.-Arch. **16**, 343–356 (1948).

The motion of a top is studied, assuming that initially the axis of symmetry is vertical, but the angular momentum vector is near the vertical. *P. Franklin.*

Nagendra Nath, N. S., and Kumar Roy, Sanat. The vibrations of an infinite linear lattice consisting of two types of particles. Proc. Indian Acad. Sci., Sect. A. **28**, 289–295 (1948).

The displacements of a linear lattice with a checkered arrangement of particles and initial disturbance are found. The method of steepest descent is used to find the asymptotic solution under a simplifying assumption on the interaction of forces. *P. Franklin* (Cambridge, Mass.).

Kašanin, R. Les équations générales du mouvement d'un système de points matériels aux liaisons données. Acad. Serbe Sci. Publ. Inst. Math. **2**, 116–130 (1948). (French. Serbian summary)

An elementary but very general derivation of the equations of motion of any dynamical system (including nonholonomic systems). D'Alembert's principle and Gauss's principle of least constraint are discussed.

D. C. Lewis (Baltimore, Md.).

Caianiello, E. Sul moto impulsivo di un sistema olonomo in presenza di vincoli unilaterali simultanei. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 706–714 (1948).

The author is concerned with a holonomic dynamical system with generalized coordinates $q_1, \cdots, q_m$ subjected to unilateral perfectly elastic constraints of the form $f_1(q_1, \cdots, q_m) \geqq 0$ and $f_2(q_1, \cdots, q_m) \geqq 0$. He is particularly interested in what happens when the moving point hits the $m-2$ dimensional manifold $f_1=0$, $f_2=0$. For the special case when $f_1=0$ and $f_2=0$ are orthogonal with respect to the action metric, he obtains a result which, in obvious notation, may be written $v_t^- = v_t^+$, $v_n^- = -v_n^+$, t referring to the tangential component of the velocity and n to the normal. *D. C. Lewis* (Baltimore, Md.).

Bilimovitch, Anton. Sur la transformation canonique des équations du mouvement d'un système non holonome. Acad. Serbe Sci. Publ. Inst. Math. **2**, 108–115 (1948). (French. Serbian summary)

A comparison between results of the author [C. R. Acad. Sci. Paris **158**, 1064–1068 (1914)] and those of L. Marchetti [Ann. Scuola Norm. Super. Pisa (2) **10**, 199–208 (1941); these Rev. **8**, 539]. Both papers treated the same problem and used the same methods with slight variations, so that it is not surprising that the results are shown to be equivalent. *D. C. Lewis* (Baltimore, Md.).

Hydrodynamics, Aerodynamics, Acoustics

Truesdell, Clifford. Généralisation de la formule de Cauchy et des théorèmes de Helmholtz au mouvement d'un milieu continu quelconque. C. R. Acad. Sci. Paris **227**, 757–759 (1948).

The author studies the change of vorticity in a fluid of the most general type, the argument being purely kinematical. The basic idea is to split the change of the vorticity vector (as we follow a fluid particle for finite time) into two parts, one of which is due to convection and the other to diffusion. The diffusion term disappears if the motion is such that the circulations in all closed circuits are conserved. In general, the convection depends only on the relative displacements of the particles at the final instant, but the diffusion depends essentially on the history of the motion. *J. L. Synge* (Dublin).

Tzénoff, Iv. Sur la déformation d'un élément infiniment petit d'un système matériel continu. C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 1, 17–20 (1948).

This paper concerns the kinematical interpretation of the rate of deformation tensor $d_{ij} = [v_{i,j} + v_{j,i}]/2$ in the motion of a continuous medium. Using a formula due to Beltrami [Opere, v. 2, 1904, pp. 202–379, in particular, § 7], the author deduces results usually attributed [e.g., Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, § 30] to Stokes but actually due to Euler [Histoire de l'Académie Royale des Sciences et Belles Lettres, Berlin **1755**, 274–315 (1757), see § XV; Novi Commentarii Academiae Scientiarium Imperialis Petropolitanae **14**, 270–386 (1770 for 1769), see §§ 9–13]. The proofs could be shortened. *C. Truesdell* (Washington, D. C.).

Huron, Roger. Sur l'unicité des solutions du problème de représentation conforme de Helmholtz (cas des obstacles polygonaux). C. R. Acad. Sci. Paris **228**, 290–292 (1949).

En prolongeant les travaux de J. Leray, J. Kravtchenko a établi un théorème d'existence pour le problème de représentation conforme de Helmholtz posé pour un obstacle tranchant, doué d'un nombre fini de points anguleux, placé dans un canal à bords rectilignes et parallèles. L'auteur complète ces résultats par une discussion d'unicité dans le cas particulier d'un obstacle polygonal. La méthode utilisée cet encore due à Leray. *J. Kravtchenko* (Grenoble).

Huron, Roger. Sur l'unicité des solutions du problème de représentation conforme de Helmholtz (cas des obstacles non lisses). C. R. Acad. Sci. Paris **228**, 357–358 (1949).

Extension des résultats de la note précédente au cas d'un obstacle formé d'arcs assez réguliers, constituant un contour à points anguleux en nombre fini. Mais cette fois les conclusions ne s'appliquent qu'aux obstacles tournant leur pointes vers le fluide vif. *J. Kravtchenko* (Grenoble).

Gregory, C. C. L. Theory of a loop revolving in air, with observations on the skin-friction. Quart. J. Mech. Appl. Math. **2**, 30–39 (1949).

Consider a belt hanging on a revolving pulley. For certain pulley speeds, taking account of air resistance, the belt becomes "air borne" and will assume a plane curve passing well above the pulley. The author analyses this situation and suggests that the results be used together with experiment to measure the frictional drag of such objects in air. The quoted results of such experiments are in good agreement with previously estimated values of air drag. *G. F. Carrier* (Providence, R. I.).

Heins, Albert E. Water waves over a channel of finite depth with a dock. Amer. J. Math. **70**, 730–748 (1948).

The problem is to discuss gravity waves of small amplitude in water of uniform finite depth when half of the water surface is covered by a rigid horizontal half-plane (the dock referred to in the title). The author finds two standing wave solutions which have such phases at ∞ that they can be combined with appropriate time factors to yield progressing wave solutions with wave crests at ∞ which are at any given angle to the edge of the dock. The similar problem for water of infinite depth, but only for the case of two-dimensional motion, has been solved by K. O. Friedrichs and H. Lewy [Communications on Appl. Math. **1**, 135–148 (1948); these Rev. **10**, 336]. The author solves the problem by reducing it to an integral equation of the Wiener-Hopf type with the aid of a Green's function for a plane strip. The integral equation turns out to be explicitly solvable. *J. J. Stoker* (New York, N. Y.).

Riegels, F. Das Umströmungsproblem bei inkompressiblen Potentialströmungen. I. Ing.-Arch. **16**, 373–376 (1948).

In the case of a symmetrical profile at no incidence, the potential problem would be solved by mapping the profile in the z-plane on a slit in a ζ-plane, and distributing sources along the slit to satisfy the boundary condition. The derivative of the transformation at points of the contour is needed to calculate the surface velocity; its value is $ds/d\xi$, where ds denotes the element of arc length of the profile and the slit lies on the ξ-axis ($\zeta = \xi + i\eta$). The author proposes to use the approximate relation $ds/d\xi = ds/dx$, c=constant, where

$z=x+iy$. This leads to simple formulas for the surface velocity distribution and seems to be a good approximation for sufficiently thin profiles. Two examples are shown.
W. R. Sears (Ithaca, N. Y.).

Luntz, Michel. Aérodynamique moléculaire. Recherche Aéronautique **1949**, no. 7, 17–33 (1949).

After a brief discussion of the cases where the conventional fluid mechanics based upon the Navier-Stokes equations is not applicable, the author gives a rather extended exposition of the elements of kinetic theory of gases. The interaction of the gas and a moving wall in the free molecular flow case is then treated, following the calculations by Sänger and Bredt, and Tsien [J. Aeronaut. Sci. **13**, 653–664 (1946); **15**, 573–580 (1948)]. The paper concludes with numerical results for the following specific examples under the assumption of complete diffuse re-emission and unit accommodation coefficient: (1) insulated flat plate without radiation heat loss; (2) heat conductive flat plate without radiation; (3) drag of a cone without radiation and (4) drag of elliptic cylinders and spheres. *H. S. Tsien.*

Krzywoblocki, M. Z. On certain cases of simple exact solutions of flow equations in a compressible imperfectly viscous fluid with particular conditions. Math. Mag. **22**, 111–123 (1949).

This paper is intended to explore cases of flow of a compressible viscous ideal gas with constant coefficient of viscosity and heat conductivity, where exact solutions, although physically unrealizable, are mathematically solvable. These cases are: two-dimensional source, circular vortex and spiral vortex. In all the cases studied, the special form of solutions makes the pressure, density, and temperature, as well as the velocity, equal to zero at infinity. In the case of the vortex, the same assumption necessitates that the temperature is negative in the whole field
Y. H. Kuo (Ithaca, N. Y.).

Ferri, Antonio. Application of the method of characteristics to supersonic rotational flow. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **841**, 12 pp. (1946).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1135 (1946); these Rev. **8**, 106.

Ferri, Antonio. The method of characteristics for the determination of supersonic flow over bodies of revolution at small angles of attack. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1809**, 53 pp. (1949).

Previous investigations on this subject have been based on linearized theory or at least on the assumption of irrotational flow. Even at small incidence this assumption leads to appreciable errors; hence it is avoided in the present paper. The method consists essentially in assuming, for small incidence α, that the velocity components and properties of state are of the form (1) $f(x, y, \theta)=f_1(x, y)+\alpha f_2(x, y)\cos\theta$, where x, y, θ are cylindrical coordinates aligned with the body. This assumption is introduced into the equations of motion and energy, assuming a perfect gas and uniform stagnation enthalpy. It is then shown how the nose shock wave is distorted by the incidence, and that the result is consistent with (1). The shock is no longer a surface of revolution; it is the tangent surface of the same cones as for zero incidence but each generating cone is inclined to an incidence proportional to α but varying from cone to cone, i.e., along the shock surface. It is shown that the characteristic surfaces of the equations of motion are also of this type.

The first-order differential equations for the velocity components u and v along the two families of characteristics are found in the usual manner [cf. the preceding review], and also expressions for the component w and the entropy. Two practical procedures are suggested for solution; one graphical-numerical, the other a numerical scheme in which the characteristic net for zero incidence is used. In an appendix a new method is given for the calculation of the flow around a circular cone at a small incidence. *W. R. Sears.*

Tsien, Hsue-Shen, and Baron, Judson R. Airfoils in slightly supersonic flow. J. Aeronaut. Sci. **16**, 55–61, 64 (1949).

Certain simple supersonic flows are approximated from the point of view of von Kármán's similarity law of transonic flow [J. Math. Physics **26**, 182–190 (1947); these Rev. **9**, 217]; namely, those associated with a straight oblique shock, a Prandtl-Meyer expansion, and the flow past aerofoils in the form of a flat plate or an isosceles triangle. Pressure, lift, drag, moment coefficients in slightly supersonic flow are plotted as functions of the von Kármán similarity parameter K and of the angle of attack, as well as in other ways. The smaller the Mach number, the smaller must be the deflection of the stream if the formulae are to be applicable. *M. J. Lighthill* (Manchester).

Reissner, Eric. On compressibility corrections for subsonic flow over bodies of revolution. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1815**, 9 pp. (1949).

Linearized subsonic flow past a corrugated cylinder is considered, and the solution is shown to involve the Bessel function $K_0(\pi\beta r/l)$, where β^2 denotes $1-M^2$, M the stream Mach number, r the radial coordinate, and $2l$ the wave length of the corrugation. This leads to a velocity-correction formula, for varying M, for such a corrugated cylinder:

$$\frac{u_s}{u_i}=\frac{1}{\beta}\,\frac{K_0(\pi\beta a/l)}{K_1(\pi\beta a/l)}\,\frac{K_1(\pi a/l)}{K_0(\pi a/l)},$$

where u_s and u_i are the axial surface velocity components at Mach numbers M and zero, respectively, and a denotes the mean radius of the cylinder. When $\pi\beta a/l\gg 1$, this reduces to the Prandtl-Glauert rule, $u_s/u_i=1/\beta$. When $\pi a/l\ll 1$, Göthert's formula is obtained [see Lees, Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1127** (1946); these Rev. **8**, 108]. The author proceeds to treat the case of a cylinder with a single bump, and obtains a result which has the form deduced by the reviewer [Quart. Appl. Math. **5**, 89–91 (1947); these Rev. **8**, 540]. Finally the author treats the case of such a cylinder in a flow confined by a circular tunnel wall. *W. R. Sears* (Ithaca, N. Y.).

Cherry, T. M. Flow of a compressible fluid about a cylinder. II. Flow with circulation. Proc. Roy. Soc. London. Ser. A. **196**, 1–31 (1949).

This is an extension, to the case where circulation is not zero, of the author's previous paper [same Proc. Ser. A. **192**, 45–79 (1947); these Rev. **9**, 544] on the flow of a compressible fluid about a cylinder. The author starts out to show that, by direct application of his method of constructing a solution for a compressible fluid to this problem, the solution obtained will be multiple-valued. To overcome this difficulty, it is necessary to add another multiple-valued function, involving two infinite sets of constants which are so chosen as to make the solution single-valued. This condition gives rise to two infinite sets of linear equations of

which special solutions are found. This solution then is a double series. This complication is necessary to eliminate the logarithmic singularity in the coordinate functions at the stagnation points in Lighthill's solutions [same Proc. Ser. A. 191, 352–369 (1947); these Rev. 9, 391]. For the complete solution, this is continued; and the analytic continuation is carried out in a quite analogous manner. By a similar procedure the stream function is also constructed. The flow pattern is not calculated. *Y. H. Kuo.*

Cherry, T. M. Numerical solutions for transonic flow. Proc. Roy. Soc. London. Ser. A. 196, 32–36 (1949).

In this paper the author presents the flow patterns past a cylinder, produced by superposition of a cosine-term solution and a sine-term solution to that generated from an incompressible flow past a cylinder without circulation. The first picture shows a flow about a distorted circular cylinder with free stream Mach number 0.510. The body is symmetrical about the coordinate axes but is flattened on top and bottom. Cases of asymmetric distortion are clearly shown in the second and third pictures; and in each of these cases the limiting-line is on the verge of appearing on the surface of the body. By this procedure, the author attempts to derive reasonably good aerodynamic body profiles.

Y. H. Kuo (Ithaca, N. Y.).

Richter, W. Eindimensionale stationäre Gleichdruckströmung in bewegten Systemen. Ing.-Arch. 16, 422–445 (1948).

In connection with studies on "constant pressure" compressors (e.g., axial compressors), the author investigates the one-dimensional compressible flow along a curve in a plane, a cylindrical surface or a surface of revolution with body force but with constant pressure. The plane is assumed either to move with a constant velocity in its plane or to rotate around an axis perpendicular to the plane. The cylindrical surface is assumed to move with a velocity parallel to its generatrix. The surface of revolution is assumed to move along its axis or rotate around its axis. The square of the absolute velocity of the fluid is assumed to vary essentially linearly with the distance from the origin for relative motion in a plane or a cylindrical surface, from the axis for relative motion in a surface of revolution. The problem is to calculate the fluid path (which is the "blade shape" in a compressor). It turns out that, by using simple transformations, all problems stated above can be reduced to the first problem: motion in a plane which is itself in linear motion. Analytical and graphical solutions are given.

H. S. Tsien (Pasadena, Calif.).

✶Riabouchinsky, D. P. Hydraulic analogy of the motion and resistance of a compressible fluid as an aid to aeronautical research. Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 61–88. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

Meyer, R. E. The method of characteristics for problems of compressible flow involving two independent variables. II. Integration along a Mach line. The radial focusing effect in axially symmetrical flow. Quart. J. Mech. Appl. Math. 1, 451–469 (1948).

In part I [same J. 1, 196–219 (1948); these Rev. 10, 338] the differential equation was derived which relates the rates of change of velocity components and pressure along a characteristic curve in plane or axisymmetric steady flow. Here the equation is differentiated with respect to the normal coordinates to obtain the equation governing the distribution of the normal derivatives of velocity components and the variables of state along the characteristics. For simplicity, isentropic irrotational flow of a perfect gas is considered, and in particular the case of straight characteristics, for which the differential equation is easily integrated. This permits a study of the "radial focusing effect" in axisymmetric flow.

The results lead to some conclusions about the maximum curvature disturbance which a supersonic diffuser may introduce at its lip without producing a shock wave. It is shown that, for given initial curvature of the lip, shockless flow is possible in a certain range of Mach numbers, providing that the curvature is not too large. Similar rules are obtained for plane flow. The similarity of the results of nonlinear and linearized theories is pointed out. As an example, the flow at the entrance to a certain diffuser is calculated by Massau's method (from part I) and also by evaluating the focusing effect on first- and second-order disturbances according to the equations of the present paper. The possibility of approximating to such a flow by these analytical expressions is discussed. Finally, the nature of the singularity at the axis is investigated with the aid of linear theory. *W. R. Sears* (Ithaca, N. Y.).

Lotkin, Mark. Supersonic flow over bodies of revolution. Quart. Appl. Math. 7, 65–74 (1949).

Equations are derived by which the steady symmetrical flow near the nose of a pointed projectile might be computed as a perturbation of the conical flow near the tip of a circular cone. The characteristic curves of the perturbation equations are those of the original conical flow. The flow involves first-order vorticity, which must be considered, due to the perturbation of the nose shock wave. The computational procedure is outlined in some detail, but no example of its use is presented. It is difficult for the reviewer to assess the value of this theory in comparison with a straightforward computation by the method of characteristics. *W. R. Sears* (Ithaca, N. Y.).

Germain, P. La théorie des mouvements homogènes et son application au calcul de certaines ailes delta en régime supersonique. Recherche Aéronautique 1949, no. 7, 3–16 (1949).

Homogeneous flow of order n is defined as that for which the perturbation potential $\varphi(x_1, x_2, x_3)$ is a homogeneous function of order n in the variables x_1, x_2, x_3. In such a case the nth derivatives of φ are constant along rays through the origin. Conical flow is obtained when $n=1$. The author considers flow past thin, nearly-plane wings; in particular, triangular (delta) wings whose cross-sections across the stream direction are affine transformations of one another, and deduces that such flows can be studied by superposition of homogeneous flows. As an example he treats the case of a wing, at zero incidence, whose spanwise cross sections are ellipses, and whose chordwise sections are described by a polynomial. Next, he treats the case in which the surface is described by a homogeneous polynomial in two variables. This constitutes a calculation of elementary cases. A number of the elementary pressure functions are worked out in detail. The thin lifting wing is then discussed, particularly the case $n=2$. The paper closes with a section devoted to the idea of solving these homogeneous-flow problems by means of an electrical analogy. [Throughout the paper,

frequent use is made of results from another report, as yet unpublished and therefore not available to the reviewer.]
W. R. Sears (Ithaca, N. Y.).

Roumieu, Charles. Étude des régimes transitoires en aérodynamique supersonique à deux dimensions. C. R. Acad. Sci. Paris **228**, 56–57 (1949).

The perturbation velocity potential associated with a thin two-dimensional obstacle is taken to obey the linear wave equation. The flow direction velocity component at the obstacle surface is written in terms of the (known) normal component by use of a well-known equation. The author states that the result reduces readily to that of Possio in the case of the oscillating obstacle. *G. F. Carrier.*

Sedov, L. I. Hydro-aerodynamical forces on a streamlined profile in a compressible fluid. Doklady Akad. Nauk SSSR (N.S.) **63**, 627–628 (1948). (Russian)

The barotropic flow past a cascade of obstacles is considered. The lift on a single member of the cascade is computed in terms of the circulation. The limit case is considered where the gap tends to infinity. The conclusion thus reached, that the Joukowski formula is valid for barotropic flows past a single airfoil, does not follow (e.g., note that the flow at ∞ is not regular in this limit case) and the question of the validity of the result remains open.
G. F. Carrier (Providence, R. I.).

Carrier, G. F. The oscillating wedge in a supersonic stream. J. Aeronaut. Sci. **16**, 150–152 (1949).

This is a preliminary report of a study of the effects on the aerodynamic characteristics produced by small oscillations of a two-dimensional wedge in an otherwise steady supersonic stream. By considering small oscillations, the modified motion can be represented by a system of acoustic waves characterized by three "wave-functions." These functions are then determined to satisfy the prescribed conditions on the wedge and on the oblique shock attached to the nose. The fact that a shock instead of a Mach wave is assumed in the unperturbed flow is expected to have a pronounced effect on the pressure distribution over the body. As the wedge executes oscillations about its leading edge, sufficiently large velocity of the wedge might cause the shock to detach. A critical condition for nondetaching shock is discussed. Presumably, the waves are generated by the oscillating body and are subsequently reflected by the shock; the reason why only downstream-moving waves were considered is not clear. *Y. H. Kuo* (Ithaca, N. Y.).

Kuznecov, P. I. The oscillations of an airfoil in a supersonic flow. Doklady Akad. Nauk SSSR (N.S.) **64**, 301–304 (1949). (Russian)

The oscillations of a thin airfoil in a supersonic stream are considered on the basis of the conventional perturbation theory. The Laplace transform analysis is used and a fundamental error in its usage leads to a result which is, in general, incorrect. *G. F. Carrier* (Providence, R. I.).

Falkovich, S. V. On the theory of the Laval nozzle. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1212**, 16 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 503–512 (1946); these Rev. **8**, 416.

Oswatitsch, Kl., and Rothstein, W. Flow pattern in a converging-diverging nozzle. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1215**, 42 pp. (1949).

Translated from Jahrbuch **1942** der Deutschen Luftfahrtforschung, pp. I95–I102; these Rev. **9**, 391.

v. d. Waerden, B. L., and Korevaar, J. Evaporation into a turbulent atmosphere. Math. Centrum Amsterdam. Rapport **ZW 1948-006**, 6 pp. (1948). (Dutch)

Let air be moving parallel to the surface of a strip of water and in a direction perpendicular to the edges of the strip. The authors propose to determine the rate at which the water evaporates due to the action of turbulence in the air stream. The motion of the air is steady and the velocity gradient is proportional to $\log (z+z_0)/z_0$, where z is the distance above the water and z_0 represents a friction coefficient at the interface. An exact solution by the Laplace transformation, similar to the solution of the heat equation, was attempted, but could not be carried through. Instead, an approximate solution, also using the Laplace transformation, is presented and the validity of the approximation discussed. *W. J. Nemerever* (Ann Arbor, Mich.).

Chandrasekhar, S. The theory of statistical and isotropic turbulence. Physical Rev. (2) **75**, 896–897 (1949).

The author obtains an explicit formula for the spectrum of turbulence at high frequencies from an equation of Heisenberg, who gave only an interpolation formula. The present formula is $F(k)=F(k_0)(k_0/k)^{5/3}[1+(k/k_s)^4]^{-4/3}$, where $F(k)$ is the spectrum and k_0 and k_s are two typical wave numbers ($k_0/k_s \ll 1$). [Reviewer's remark. Essentially the same result is said to have been obtained independently by J. Bass.] *C. C. Lin* (Cambridge, Mass.).

von Kármán, Theodore. Progress in the statistical theory of turbulence. J. Marine Research **7**, 252–264 (1948).

***Robbertse, W. P. Oor die Verdigtings- en Verdunningsverskynsels in 'n Gas Veroorsaak Deur die Stoot van 'n Suier met 'n Baie Hoë Snelheid. [On the Compression and Expansion Phenomena in a Gas Caused by a Collision of a Piston Moving With a Velocity Much Exceeding That of Sound].** N. V. Noord-Hollandsche Uitgevers Maatschappij, Amsterdam, 1948. vi+104 pp. (Dutch. English summary)

A semi-infinite cylindrical tube, closed at one end by a movable piston, is filled with an ideal gas. The piston suddenly acquires a high supersonic velocity. The author investigates the subsequent motion of the gas and of the piston. This problem was first considered by J. M. Burgers [see Nederl. Akad. Wetensch., Proc. **50**, 262–271, 332–338, 442–451 (1947); these Rev. **8**, 606, 607; also Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 476–484, 560–570 (1943); these Rev. **7**, 500]. The equation of motion of the gas is derived in Langrangean form together with the two boundary conditions; the first deals with the piston, while the second takes into account the compression wave created by the sudden motion of the piston. The initial density of the gas is assumed to vary along the tube and the gas has an initial velocity also varying along the tube but which is small compared to the piston velocity.

Burgers has obtained a solution of the nonlinear, second order, partial differential equation of motion in the form of a power series in t and s, where t refers to the time and s is a parameter identifying the various layers of the gas. This solution, which satisfies the differential equation and the

boundary conditions exactly, converges only for small values of t. Since this method fails for large values of t and since great interest attaches to the solution for large values of t a different method of solution had to be used, namely separation of variables. By this method an exact solution was obtained only for the case where the gas density varies in a particular manner along the tube. Two types of approximate solutions were also obtained: either the differential equation was satisfied exactly and the boundary conditions approximately or the other way around. In the former case the solution depends on a parameter which can be so chosen as to minimize the discrepancy from the boundary conditions. A partially successful attempt was also made to introduce correction factors in one of the approximate solutions. The calculations involved in the various solutions and the results obtained are presented in great detail.

W. J. Nemerever (Ann Arbor, Mich.).

Thomas, T. Y. Calculation of the curvatures of attached shock waves. J. Math. Physics 27, 279–297 (1949).

In an earlier paper [same J. **26**, 62–68 (1947); these Rev. **8**, 611] the author derived a formula for the ratio K/k of the streamline curvature just behind a shock, and the shock wave curvature at the same point. In this paper the value of K/k is computed at the vertex of a pointed symmetric obstacle in a supersonic 2-dimensional flow with shock wave attached. At this particular point, of course, K is simply the curvature of the obstacle contour at its vertex. The parameters involved are the free stream Mach number M and the angle α between obstacle contour and shock. The semi-apex-angle ω of the obstacle is likewise given in terms of the same parameters.

A series (for different values of M) of tables and graphs are given for K/k and ω against α. Two special values of α (with the corresponding values of ω) are of interest. One is $\alpha(M)$ (and $\omega(M)$), the value of α at which ω assumes its maximum. The other, $\alpha_s(M)$ (in all cases just slightly less than $\alpha(M)$), the "singular" value of α for the given Mach number (and $\omega_s(M)$), is the value at which K/k vanishes and becomes negative. A recent paper [Proc. Nat. Acad. Sci. U. S. A. **34**, 526–530 (1948); these Rev. **10**, 271] not referred to here discusses from a somewhat different point of view the critical nature of $\alpha_s(M)$ with respect to the stability of the two shock configurations corresponding to a given value of ω. Graphs against M of these special values of α and ω are included.

D. P. Ling (Murray Hill, N. J.).

Dugundji, John. An investigation of the detached shock in front of a body of revolution. J. Aeronaut. Sci. **15**, 699–705 (1948).

A detached shock and a blunt body are a distance δ apart along the axis of symmetry. (The two-dimensional and the axially symmetric case are both included in the discussion.) Using the flow equations and the shock conditions, the stream velocity u on the symmetry axis is expanded in a power series around the nose of the shock in terms of distance along the symmetry axis. When the velocity so obtained is required to vanish at the nose of the body (assuming δ small) an expression is obtained for δk, where k is the curvature of the shock at its nose. The linear and quadratic approximations for δk are functions of free stream Mach number alone; the cubic approximation involves in addition the curvature k_0 of the body at its nose. Measurements on photographs of spheres in flight furnish several experimental values of δk, and these are compared with the theoretical. The agreement, while not altogether satisfactory, suggests that further extension of this scheme to a higher order of approximation would be expected to give accurate results concerning flow conditions along this portion of the symmetry axis.

D. P. Ling.

Busemann, Adolf. A review of analytical methods for the treatment of flows with detached shocks. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1858**, 23 pp. (1949).

Bordoni, Piero Giorgio, and Gross, Wolf. Sound radiation from a finite cylinder. J. Math. Physics **27**, 241–252 (1949).

The authors present a method for approximately calculating the radiation of a vibrating body of any shape. The method is applied to a cylindrical box with diameter equal to height. The normal gradient of the velocity potential is taken constant over one of the circular faces while the remainder of the box is assumed perfectly rigid (model of back-enclosed loudspeaker). The solution to Neumann's external boundary-value problem for the wave equation is approximated as follows. Choosing the centre and the axis of the box as origin and polar axis, respectively, of a system of spherical polar coordinates, the authors postulate the actual sound field as an infinite series of spherical wave functions, viz.,

$$(1)\qquad \varphi=r^{-1}e^{-jkr}\sum_{n=0}^{\infty}(x_n+jy_n)P_n(\cos\theta)f_n(jkr),$$

with undetermined coefficients x_n+jy_n. First, the normal component of grad φ on the surface of the box is calculated by using $p+1$ terms of (1). Second, the absolute difference between the calculated and given values is squared and then integrated over the surface of the box, leading to a quadratic form Q. Third, the coefficients x_n+jy_n ($n=0, 1, \cdots, p$) are determined so as to minimize Q. Numerical calculations are worked out for a practical example, and radiation patterns are shown and compared with those of an equivalent spherical box, using five terms ($p=4$) of (1). It is suggested that as $p\to\infty$ the procedure leads to the rigorous solution of Neumann's problem for the box. [The authors' proof of this theorem is incomplete; it would at least be necessary to show that the minimum of Q tends to zero as $p\to\infty$. Moreover, it seems to the reviewer that the series (1) is divergent on the surface of the box, which makes the validity of the theorem in question quite problematical.]

C. J. Bouwkamp (Eindhoven).

Braumann, Hans. Mediumrückwirkung und akustische Strahlungsdämpfung für ein kreisförmiges Plättchen. Z. Naturforschung **3a**, 340–350 (1948).

The author studies the acoustic radiation of the freely vibrating circular disk. The wave equation is separated in oblate spheroidal coordinates [the author terms them bipolar coordinates]. He derives approximate expressions for the inertia increase and the energy radiation of the disk. [Actually these are the first terms of the rigorous expressions developed in ascending powers of the wave number.] Both the author's method and his results are not new; more accurate results are available. The author is apparently unacquainted with the work of, e.g., Kotani [Proc. Phys.-Math. Soc. Japan (3) **15**, 30–57 (1933)], Hanson [Philos. Trans. Roy. Soc. London. Ser. A. **232**, 223–283 (1933)], King [Proc. Roy. Soc. London. Ser. A. **153**, 1–16, 17–40 (1935)], Bouwkamp [Groningen thesis, 1941; these Rev. **8**,

179] and Sommerfeld [Ann. Physik (5) 42, 389–420 (1943); these Rev. 5, 121]. *C. J. Bouwkamp* (Eindhoven).

Miles, John W. The diffraction of a plane wave through a grating. Quart. Appl. Math. 7, 45–64 (1949).

The problem discussed is the diffraction of a normally incident plane wave of sound by an infinite plane grating of infinitely thin, coplanar, equally spaced, parallel strips. The velocity potential on either side of the grating is written as a Fourier series in terms of the aperture values, and an integral equation for the aperture field is derived. A solution to this integral equation is given that reduces the problem to the solution of an infinite set of linear equations, which are solved approximately. The author gives several alternative formulations of his problem, including variational principles following J. S. Schwinger. The case of electromagnetic plane waves is treated also, and Babinet's principle [cf. Copson, Proc. Roy. Soc. London. Ser. A. 186, 100–118 (1946); these Rev. 8, 179] is discussed. Results of physical importance are given in the form of graphs. The integral equations for an aperture of finite thickness are set up, and an approximate solution including only first-order terms in thickness/wavelength is given. *C. J. Bouwkamp.*

Elasticity, Plasticity

Weber, C. Spannungsfunktionen des dreidimensionalen Kontinuums. Z. Angew. Math. Mech. 28, 193–197 (1948). (German. Russian summary)

The author investigates two sets of stress functions F_{ij} ($i, j=1, 2, 3$) which arise from the stress equations of equilibrium without body forces. These functions are related to those given by Love [A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1927] as follows:

$$F_{11}=\chi_1, \quad F_{22}=\chi_2, \quad F_{33}=\chi_3,$$
$$F_{12}=-\tfrac{1}{2}\psi_3, \quad F_{23}=-\tfrac{1}{2}\psi_1, \quad F_{31}=-\tfrac{1}{2}\psi_2.$$

It is shown that if the stresses τ_{gh} are expressed in terms of any set of stress functions in the form $\tau_{gh}=\sum_{i,k}\partial^2F_{ik}^{(gh)}/\partial x_i\partial x_k$ these stress functions are essentially the same as those given above. The array of stress functions F_{ij}, where $F_{ij}=F_{ji}$, constitute a tensor. Furthermore, the differences of any two sets of stress functions which yield the same state of stress satisfy the compatibility equations for small strains. [Reviewer's note: the last result is indicated briefly in the section in Love cited above.] *G. H. Handelman.*

Fichera, Gaetano. Sull'equilibrio di un corpo elastico, isotropo e omogeneo. Rend. Sem. Mat. Univ. Padova 17, 9–28 (1948).

Proofs of results announced previously [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 403–408 (1947); these Rev. 9, 164]. *F. G. Dressel* (Durham, N. C.).

Prager, William. Recent developments in the mathematical theory of plasticity. J. Appl. Phys. 20, 235–241 (1949).

Survey article.

Dorn, John E. Stress-strain rate relations for anisotropic plastic flow. J. Appl. Phys. 20, 15–20 (1949).

A particular relation, $\dot{\epsilon}_{ij}=\acute{A}_{ijkl}\tau_{kl}$, between the components of strain rate ϵ and stress τ is specialized for plane stress and several types of symmetry including planar and complete isotropy. A generalized stress σ, in terms of the τ_{ij} only, and a strain rate $\dot{\varphi}$, in terms of strain only, are defined by $\dot{W}$, the rate of doing work: $\dot{W}=\sigma\dot{\varphi}$. These quantities reduce in the isotropic case to those commonly used. As the author emphasizes, he does not consider true time effects in work-hardening metals. The term strain rate is to be understood as simply the rate of increase of strain with any parameter which conveniently specifies the extent of the deformation. *D. C. Drucker* (Providence, R. I.).

Sobrero, Luigi. Sul comportamento dei sistemi elastici piani nell'intorno di spigoli rientranti. Rend. Sem. Fac. Sci. Univ. Cagliari 17 (1947), 67–87 (1948).

The author is concerned with the behavior of the stresses near the vertex in a plane elastic system containing a corner, particularly in the case when the vertex angle is greater than 180°. The author finds series expansions for Airy's stress function F such that on the infinite wedge $\theta=\pm\alpha$ the conditions $F=0$ and $\partial F/\partial\theta=0$ are satisfied. Determination of the characteristic values involves a study of the equations $z^{-1}\sin z=\pm(2\alpha)^{-1}\sin 2\alpha$. The author determines the roots of these equations having smallest positive real part. From the results of this investigation the author shows that for a vertex angle $2\alpha<180°$ the stresses vanish at the corner but that for a vertex angle $2\alpha>180°$ they generally become infinite. He proves further that if $270°\leqq 2\alpha<360°$, the stresses vary along a radius vector as the $-\frac{1}{2}$ power of the distance from the vertex, without sensible error. The author then calculates the behavior of the lines of constant principal normal and tangential stresses, sketching them for the case $2\alpha=270°$. He states that these theoretical results are fully confirmed by photoelastic experiments. *C. Truesdell* (Washington, D. C.).

Galin, L. A. An analogy for the plane elastic-plastic problem. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 757–760 (1948). (Russian)

The author points out that the values of the Airy stress function in certain problems of plane elastic-plastic strain may be obtained experimentally as the deflections of an elastic plate which is loaded along its contour by appropriate forces and couples, the deflections being limited by certain rigid surfaces which represent the statically determinate stress functions in the plastic regions. This analogy resembles the well-known soap film and sand hill analogies for elastic-plastic torsion. [For the special case of a tension strip with a symmetrically located circular hole it was indicated by the reviewer [Theory of Plasticity, Brown University lecture notes, Providence, R. I., 1942, p. 148]. The analogy is based, however, on a tacit assumption which has still to be investigated critically: the stress function φ for the plastic region is formed by integrating the hyperbolic equations of plane plastic equilibrium starting with the given stresses on the contour. The value of φ which is so obtained at a generic point P is correct only if it can be shown that P cannot become plastic before its domain of dependence (arc of the boundary intercepted by the two characteristics through P) has become plastic.] *W. Prager* (Providence, R. I.).

Winzer, Alice, and Carrier, G. F. The interaction of discontinuity surfaces in plastic fields of stress. J. Appl. Mech. 15, 261–264 (1948).

Discontinuity surfaces or "shocks" in the plane-strain plasticity problems, which were first introduced by W.

Prager [Courant Anniversary Volume, pp. 289–300, New York, 1948; these Rev. 10, 82], are studied with respect to the manner in which they may intersect when they separate fields of constant stress. The authors find that the least number of shocks, separating states of constant stress, which form a consistent system is four. The results are applied to a trapezoid subject to uniform loads applied to the parallel sides. It is found that the load necessary to produce this solution is less than that required by the Prandtl solution [Z. Angew. Math. Mech. 1, 15–20 (1921)]. It is shown, in an analysis of the interaction of shock and "fan fields," that no fewer than three shocks can intersect at the vertex of a fan. The authors also show that in the case where a shock intersects a stress-free surface, the angle through which the "stream" must turn and the angle of the reflected shock are defined uniquely. The results are also extended to the case where one of the surfaces of the body supports a uniform normal load. *G. H. Handelman.*

Sokolovskiĭ, V. V. Plane limiting equilibrium of geological strata. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1361–1370 (1948). (Russian)

Two-dimensional states of limiting equilibrium in soils are discussed under the assumption that the shearing stress in the slip planes is a function of the normal stress on these planes that is characteristic for the soil under consideration. While the problem is formulated in fairly general terms, the examples treated in the paper are rather elementary.
W. Prager (Providence, R. I.).

Hencky, H. Über die Berücksichtigung der Schubverzerrung in ebenen Platten. Ing.-Arch. **16**, 72–76 (1947).

A consideration of the effect of shear leads to three differential equations for the three components of displacement of points in an isotropic thick bent plate whose middle surface is unstretched. The procedure differs from that of E. Reissner [J. Appl. Mech. **12**, A-69–A-77 (1945); Quart. Appl. Math. **5**, 55–68 (1947); these Rev. **7**, 42; **8**, 547] but the results are somewhat similar although not carried through to explicit solutions. Displacement expressions $u=-(z/h)\psi(x,y)$, $v=-(z/h)\chi(x,y)$, $w=\varphi(x,y)$ are employed. The strains and then the stresses are calculated as though $\sigma_z=0$ but σ_z is retained in the equations of equilibrium. The principle of virtual work is then applied so that the previously found uniform distribution of transverse shear through the thickness can be taken as a particular average. The conclusion is drawn that the usual thin plate result is correct for a uniformly loaded clamped plate but the boundary stresses are too low. *D. C. Drucker.*

Wittrick, W. H., and Howard, W. Relaxation methods applied to two problems of two-dimensional stress distribution involving mixed boundary conditions. Australian J. Sci. Research. Ser. A. **1**, 135–160 (1948).

Relaxation methods have been used to determine the stress distributions in both a rectangular and a highly tapered plate under tension when the load is applied through absolutely rigid clamps. Both problems require the treatment of boundary conditions involving the values of both stresses and displacements. The solutions were obtained in terms of displacements and the stresses were subsequently determined from them. *Authors' summary.*

Pozzati, Piero. La lastra rettangolare sostenuta da un reticolo di travi di qualunque rigidezza. Boll. Un. Mat. Ital. (3) **3**, 236–248 (1948).

Stein, Manuel, and Fralich, Robert W. Critical shear stress of infinitely long, simply supported plate with transverse stiffeners. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1851**, 39 pp. (1949).

A theoretical solution is given for the critical shear stress of an infinitely long, simply supported, flat plate with identical, equally spaced, transverse stiffeners of zero torsional stiffness. Results are obtained by means of the Lagrangian multiplier method and are presented in the form of design charts. *From the authors' summary.*

✶Schnadel, Georg. The strength of transversely stiffened decks of ships. Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 256–267. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

Klitchieff, J. M. On the stability of the deck plates of steel ships. Acad. Serbe Sci. Publ. Inst. Math. **2**, 53–78 (1948). (Russian. Serbian summary)

Klitchieff, J. M. On the torsion of a rectangular tube. Acad. Serbe Sci. Publ. Inst. Math. **1**, 58–61 (1947). (Russian)

✶Odqvist, Folke K. G. Plasticity applied to the theory of thin shells and pressure vessels. Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 449–460. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The paper is concerned with the stresses and strains in a thin closed shell of revolution which is made of an elastic-perfectly-plastic material and subjected to interior pressure. Within the framework of the membrane theory of shells the stress distribution is statically determinate and hence independent of the mechanical behavior of the material. If these membrane stresses should be found to violate the yield condition anywhere, the shell would not be safe. However, in zones where the membrane theory of shells breaks down (parallels where the meridian curvature changes discontinuously), bending stresses may reach the yield limit. This problem is treated for the special case where the cylindrical part of a pressure vessel meets the conical top or bottom. [A somewhat similar analysis was given by A. A. Ilyushin, Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 15–24 (1944); these Rev. **6**, 140, who considered a strain-hardening material, however.] *W. Prager.*

Tester, K. G. Beitrag zur Berechnung der hyperbolischen Paraboloidschale. Ing.-Arch. **16**, 39–44 (1947).

The membrane stresses in a surface of the form $z=xy/n$, where n is a constant, are determined for various conditions of loading which are of interest to the structural engineer. A brief discussion points out the several advantages of this nondevelopable surface, when used in such structures as wide-span roofings (with the z-axis vertical), over cylindrical and other doubly-curved surfaces. For the particular surface studied, the expressions for the membrane stresses are obtained from the conventional theory in simple and convenient forms. In the present paper, the solutions for a wide variety of loading conditions are derived, involving various kinds of dead load distributions, as well as wind loads.
M. Goland (Kansas City, Mo.).

Ghosh, S. On the torsion and flexure of a beam whose cross-section is a quadrant of a given area. Bull. Calcutta Math. Soc. **40**, 107–115 (1948).

Utilizing Schwarz's principle of reflection, the author solves the torsion and flexure of a beam whose section has

two orthogonal axes of symmetry by an analytical continuation of the complex torsion and flexure functions to the other quadrants of the beam. The given domain is then mapped by $z=\omega(\zeta)$ upon the unit circle in the ζ plane and the complex functions are determined by the help of Schwarz's formula for the prescribed boundary values. The torsion and flexure moment expressions are constructed and applied to a beam whose section is a quadrant of a circle. The center of flexure for this section is given.

D. L. Holl (Ames, Iowa).

Swida, W. Die elastisch-plastische Biegung des krummen Stabes. Ing.-Arch. **16**, 357–372 (1948).

The author discusses first the elastic-plastic flexure of a naturally curved bar with constant radius of curvature and rectangular cross section. It is found that even when the radius of curvature is of the same order of magnitude as the height of the cross section, the greatest radial stress is considerably smaller than the greatest circumferential stress. The author then proceeds to develop an approximate theory of elastic-plastic bending of naturally curved bars in which the transverse (radial) stresses are neglected. It is worth noting that the bending moment which produces collapse according to this theory is independent of the radius of curvature of the bar.

W. Prager (Providence, R. I.).

Vogel, Théodore. Sur l'application de la théorie des équations différentielles linéaires à coefficients périodiques aux problèmes d'appuis élastiques. C. R. Acad. Sci. Paris **228**, 162–163 (1949).

Floquet's theorem is applied to the fourth order equation for a rod.

P. Franklin (Cambridge, Mass.).

✱**Gran Olsson, R. Remarks on the deflection theory of suspension bridges.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 211–230. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

Dirac, G. A. The vibration of propeller blades. An analysis of the problems of lateral torsional and longitudinal vibration developed from the simple case of a bar clamped at one end. Aircraft Engrg. **20**, 322–329, 343 (1948).

The author considers the torsional and bending vibrations of a bar, clamped at one end, which rotates with uniform velocity about a fixed axis through the clamped end, perpendicular to the axis of the bar. The rotation does not interfere with the torsional vibrations, but contributes an additional bending moment to the bending vibrations. The relevant characteristics of the bar and the torsional and lateral deviations θ, y and z are assumed to be expanded in power-series in terms of the distance x from the clamped end, for example, $\theta=\sum_{i=1}^{\infty} p_i x^i$, where $p_i=p_{i0}\operatorname{cis}(\lambda t+\tau)$. The torsional moment in a section is found as an integral over the moment of the inertia-forces outside the section. From the well-known equation $\partial\theta/\partial x=M/(\mu k)$ the author finds an infinite set of equations in the infinity of unknowns p_i. Then λ^2 are the characteristic values of infinite determinant of the set and $p_1p_2p_3\cdots$ the proper columns. The same method is used for the bending vibrations, where $y=\sum_{i=2}^{\infty} q_i x^i$ and $z=\sum_{i=2}^{\infty} r_i q^i$ ($q_i=q_{i0}\operatorname{cis}(\lambda t+\sigma)$, and $r_i=r_{i0}\operatorname{cis}(\lambda t+\chi)$) are the deflections in perpendicular directions. The proper column is here $q_2r_2q_3r_3\cdots$. The author does not mention that his formulae are only valid for $\sigma=\chi$. To calculate λ and the modes, a finite number N of the rows and columns is taken, and criteria for rapid convergence of the characteristic values and proper columns of these finite determinants to the characteristic values and proper columns of the infinite determinant are given without proof. Finally the author calculates by the same method the response to periodic forces and couples.

W. H. Muller (Amsterdam).

MATHEMATICAL PHYSICS

Quantum Mechanics

de Broglie, Louis. La statistique des cas purs en mécanique ondulatoire et l'interférence des probabilités. Revue Sci. **86**, 259–264 (1948).

A careful exposition is given of the nature of the similarities and differences between the types of probability which arise in ordinary statistics and quantum mechanics, the emphasis being on the differences. Several simple examples are worked out in some detail, one example being that of a deck of cards in which an observation of the value (number, or rank for a face card) results in a random change in the suit of the card, and vice versa, so that analysis of the situation requires considerations similar to, though not at all isomorphic with, those arising in quantum mechanics. The main arguments are to the effect that though it is possible to define conditional probabilities in quantum mechanics (in some circumstances), this cannot be done in such a way that the usual rules for compounding probabilities are valid. In particular, in the case of a stationary particle on a straight line in a pure state, a computation is given to show that it is impossible to find a function of position and momentum which would be appropriate as a joint distribution, in the sense that the usual formula for computing marginal from joint distributions is valid, or to find conditional probability functions which will validate the usual formulas for compounding these.

A secondary point (not illustrable by the case of the deck of cards), to which the author attributes invariant significance, is that in quantum mechanics a probability is always the square of a complex number. The phrase "interference of probabilities" in the title refers to the fact that the results of compounding the probabilities may depend on the arguments of these complex numbers, and so are not functions of the probability alone.

I. E. Segal (Chicago, Ill.).

Hu, N. Further investigations on Heisenberg's characteristic matrix. Proc. Roy. Irish Acad. Sect. A. **52**, 51–68 (1948).

For a system in which the number of particles is fixed, if the S-matrix is an analytic function of the momentum variables, the closed states of the system can be obtained by analytic continuation of the S-matrix to complex values of the momentum variables. If the S-matrix contains elements for transitions to states with a different number of particles, however, S cannot be continued analytically into the region in which the energy is less than the rest energy of the highest particle considered, without modifying the original S-matrix in the scattering region. Such a modification is developed, but it is shown that it leads to the result that no permanent closed states of the system are possible.

H. C. Corben (Pittsburgh, Pa.).

Sokolow, A. A. On the classical theory of elementary particles (point electron). Vestnik Moskov. Univ. **1947**, no. 2, 33–48 (1947). (Russian. English summary)

The equation of motion of a classical point electron, including the radiation-reaction term, is derived by a procedure essentially the same as that used by P. A. M. Dirac [Proc. Roy. Soc. London. Ser. A. **167**, 148–169 (1938)] and G. Wentzel [Z. Physik **86**, 479–494 (1933)]. The author does not refer to the recent and more profound analysis of this procedure by J. A. Wheeler and R. P. Feynman [Rev. Modern Physics **17**, 157–181 (1945)]. He includes a brief discussion of the problem of fixing boundary conditions so as to exclude the nonphysical self-accelerating solutions of the equation of motion, and also derives the formulae for the emission and scattering of light by an electron.

F. J. Dyson (London).

Ivanenko, D. D., and Sokolov, A. A. Quantum theory of gravitation. Vestnik Moskov. Univ. **1947**, no. 8, 103–113 (1947). (Russian)

The paper begins with a simplified derivation of the equations of M. Fierz and W. Pauli [Proc. Roy. Soc. London. Ser. A. **173**, 211–232 (1939); these Rev. **1**, 190] describing a quantized field of tensor character. Such a field can be identified with the Einstein gravitational field, in a weak-field approximation neglecting quadratic terms in the field equations. The quanta of the field, "gravitons," are particles of spin two. The intensity of gravitational radiation from a moving source of the field is calculated by perturbation theory, and is found to agree with the classically calculated intensity in the low-frequency limit. Finally, there is considered the problem of a quantized scalar field acting as the source of the gravitational field; in this case there exists a process of annihilation of a scalar particle and anti-particle, with the emission of two gravitons; also there exists an inverse process of creation. The cross-section for the annihilation process is found to be of the order of the square of the gravitational radius of a scalar particle.

F. J. Dyson (London).

Shirokov, M. F. On the role of gravitation in the structure of elementary particles. Vestnik Moskov. Univ. **1947**, no. 4, 67–75 (1947). (Russian. English summary)

The author investigates the static symmetrical solutions of the field equations of Bopp [Ann. Physik (5) **38**, 345–384 (1940); these Rev. **2**, 336] and of Born and Infeld [Proc. Roy. Soc. London. Ser. A. **144**, 425–451 (1934)], which solutions are possible models for a classical electron. Considering such an electron as a gravitating body in the general theory of relativity, he calculates the gravitational fields associated with it. He finds that the field is (a) of the Schwartzschild type at moderate distances, (b) singular at the centre of the electron. Conclusion: in discussing the nature of the singularity at the centre of these solutions, it is not permissible to neglect general relativistic effects.

F. J. Dyson (London).

Heitler, W. On the relativistic interaction of point particles. Proc. Roy. Irish Acad. Sect. A. **52**, 95–108 (1948).

The theory developed earlier [W. Heitler and H. W. Peng, Proc. Cambridge Philos. Soc. **38**, 296–312 (1942)] in which the divergent parts in the quantum theory of fields may be eliminated by a relativistically invariant subtraction procedure, is applied to the problem of the interaction between two particles. Even for the general case in which the interaction occurs through a charged field with quanta of finite rest mass it is shown that at low relative energies it is possible to define a mutual potential for the particles and at high energies it is possible to establish an invariant theory of collisions, but it is not possible in this way to develop a generally invariant divergence-free theory which gives an ordinary potential at low energies.

H. C. Corben (Pittsburgh, Pa.).

Chang, T. S. Relativistic field theories. Physical Rev. (2) **75**, 967–971 (1949).

A general formulation is given of the change of Schrödinger wave functionals on space-like surfaces as the surface changes, and the condition is imposed that the expectation value of a field variable at a point is independent of the choice of surface through the point or of choice of coordinates which specify the surface. When a Lagrangian principle is applied, the relation to Weiss's formalism [Proc. Roy. Soc. London. Ser. A. **156**, 192–220 (1936)] is established, and by a transformation on the wave functionals and observables the formalism of Tomonaga is obtained. It is therefore inferred that the theory of Tomonaga is essentially the same as the original Heisenberg-Pauli formalism and that it will encounter the same difficulties.

H. C. Corben (Pittsburgh, Pa.).

Jauch, J. M. On the relativistic invariance of the canonical field equations and the location of energy, momentum and angular momentum in a field. Anais Acad. Brasil. Ci. **20**, 353–361 (1948).

The general formulation of the quantum theory of fields is given, the theory of canonical transformations and their relation to Lorentz transformations explained and the method of introducing momentum-, energy-, and angular momentum-densities is described.

H. C. Corben.

Tonnelat, Marie-Antoinette. Théorie unitaire du champ physique. II. Cas d'une métrique symétrique. C. R. Acad. Sci. Paris **228**, 660–662 (1949).

In a previous paper [same vol., 368–370 (1949); these Rev. **10**, 408] the author derived a set of equations (I) which determine an arbitrary affine connection from a Lagrangian principle. In this paper the author assumes that there is a metric in the space over which the Lagrangian is integrated. The equations (I) are broken up into two sets, the symmetric set and the anti-symmetric one. The latter are solved explicitly for simple linear combinations of the coefficients of torsion as functions of certain derivatives of the Lagrangian function.

A. H. Taub (Urbana, Ill.).

Wentzel, Gregor. Über die Feldgleichungen in quantisierten Feldtheorien. Z. Naturforschung **3a**, 430–434 (1948).

Using the Tomonaga equation, the general equations in the quantum theory of fields are written in a relativistically invariant form and the renormalization of the mass of a scalar particle by means of a unitary transformation is discussed.

H. C. Corben (Pittsburgh, Pa.).

Sérgio, Paulo. Representation of variable spin. Anais Acad. Brasil. Ci. **20**, 261–271 (1948). (Portuguese)

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